Chapter 7 Solutions to Selected Exercises

Notes:

- The questions are in a separate PDF on LongFormMath.com.
- For most problems there are many correct solutions, so the below are not the only correct ways to solve the problems.
- If you spot an error, please email it to me at LongFormMath@gmail.com. Thanks!

Solution to Question 1. Let $f : A \to \mathbb{R}$ and $c \in A$. We say f is *differentiable* at c if

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

exists. That is, this limit equals some real number.

If C is the collection of points at which f is differentiable, then the *derivative* of f is the function $f' : C \to \mathbb{R}$ where

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}.$$

□

Solution to Question 2. Part (a). Let I be an interval and $f : I \to \mathbb{R}$ be given by $f(x) = x^2 + 3x + 7$. Then for any $c \in I$,

$$\begin{aligned} f'(c) &= \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \\ &= \lim_{x \to c} \frac{(x^2 + 3x + 7) - (c^2 + 3c + 7)}{x - c} \\ &= \lim_{x \to c} \frac{(x^2 - c^2) + 3(x - c)}{x - c} \\ &= \lim_{x \to c} \frac{(x - c)(x + c) + 3(x - c)}{x - c} \\ &= \lim_{x \to c} [(x + c) + 3] \end{aligned}$$

Recall that since x is a variable and c is a constant, the function $(x+c)+3$ is continuous, so we can substitute $x = c$.

$$\begin{aligned} &= (c + c) + 3 \\ &= 2c + 3. \end{aligned}$$

That's the derivative at every fixed c, so we can instead rewrite this using the variable x: $f'(x) = 2x+3$. Or, writing it out as the derivative of a function:

$$\frac{d}{dx}(x^2 + 3x + 7) = 2x + 3.$$

□

Part (b). Let I be an interval and $g : I \to \mathbb{R}$ be given by $g(x) = \frac{1}{x}$. Then for any $c \in I$ (with $c \neq 0$),

$$\begin{aligned} g'(c) &= \lim_{x\to c} \frac{g(x)-g(c)}{x-c} \\ &= \lim_{x\to c} \frac{\frac{1}{x}-\frac{1}{c}}{x-c} \\ &= \lim_{x\to c} \frac{\frac{c-x}{cx}}{x-c} \\ &= \lim_{x\to c} \frac{c-x}{cx(x-c)} \\ &= \lim_{x\to c} \frac{-(x-c)}{cx(x-c)} \\ &= \lim_{x\to c} \left(-\frac{1}{cx}\right) \end{aligned}$$

Again, the function $-\frac{1}{cx}$ is continuous in x (for $x \neq 0$), so we substitute $x = c$.

$$\begin{aligned} &= -\frac{1}{c\cdot c} \\ &= -\frac{1}{c^2}. \end{aligned}$$

That's the derivative at every fixed c, so we can instead rewrite this using the variable x: $g'(x) = -\frac{1}{x^2}$. Or, writing it out as the derivative of a function:

$$\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}.$$

□

Solution of Question 3. Let $f, g : (a, b) \to \mathbb{R}$ be differentiable at $c \in (a, b)$, and let $\alpha, \beta \in \mathbb{R}$. Define $h(x) = \alpha f(x) + \beta g(x)$. Then for any $c \in (a, b)$,

$$\begin{aligned} h'(c) &= \lim_{x\to c} \frac{h(x)-h(c)}{x-c} \\ &= \lim_{x\to c} \frac{\alpha f(x)+\beta g(x) - (\alpha f(c)+\beta g(c))}{x-c} \\ &= \lim_{x\to c} \frac{\alpha\big(f(x)-f(c)\big)+\beta\big(g(x)-g(c)\big)}{x-c} \\ &= \lim_{x\to c} \left[\alpha\frac{f(x)-f(c)}{x-c} + \beta\frac{g(x)-g(c)}{x-c}\right] \\ &= \alpha \lim_{x\to c} \frac{f(x)-f(c)}{x-c} + \beta \lim_{x\to c} \frac{g(x)-g(c)}{x-c} \\ &= \alpha f'(c) + \beta g'(c). \end{aligned}$$

That's the derivative at every fixed c, so we can instead rewrite this using the variable x: $(\alpha f + \beta g)'(x) = \alpha f'(x) + \beta g'(x)$. Or, writing it out as the derivative of a function:

$$\frac{d}{dx}(\alpha f + \beta g) = \alpha f' + \beta g'.$$

□

Solution of Question 4. By the definition of the derivative, if the following limit exists then

$$f'(0) = \lim_{x\to 0} \frac{f(x)-f(0)}{x-0}.$$

We will show that this limit does not in fact exist, which would show that f is not differentiable at 0.

Recall that a functional limit exists if and only if all of the corresponding sequential limits also exist, and moreover converge to the same value. Consider a sequence a_n such that $a_n \to 0$ and $a_n \in \mathbb{Q}$ for all n; certainly such a sequence exists (eg. $a_n = 1/n$). Then

$$\lim_{n\to\infty} \frac{f(a_n) - f(0)}{a_n - 0} = \lim_{n\to\infty} \frac{\frac{1}{2}a_n - \frac{1}{2}\cdot 0}{a_n - 0} = \lim_{n\to\infty} \frac{\frac{1}{2}a_n}{a_n} = \lim_{n\to\infty} \frac{1}{2} = \frac{1}{2}.$$

Now, however, consider a sequence b_n such that $b_n \to 0$ and $b_n \notin \mathbb{Q}$ for all n; certainly such a sequence exists (eg. $b_n = \sqrt{2}/n$). Then

$$\lim_{n\to\infty} \frac{f(b_n) - f(0)}{b_n - 0} = \lim_{n\to\infty} \frac{b_n - \frac{1}{2}\cdot 0}{b_n - 0} = \lim_{n\to\infty} \frac{b_n}{b_n} = \lim_{n\to\infty} 1 = 1.$$

□

Solution to Question 5.

- Let I be an interval and let $f, g : I \to \mathbb{R}$ be differentiable at $c \in I$. Then the product rule says that

$$(fg)'(c) = f'(c)g(c) + f(c)g'(c).$$

- Let I be an interval and let $f, g : I \to \mathbb{R}$ be differentiable at $c \in I$. If also $g(c) \neq 0$, then the quotient rule says that

$$\left(\frac{f}{g}\right)'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{\big[g(c)\big]^2}.$$

- Let I_1 and I_2 be intervals and $g : I_1 \to I_2$ and $f : I_2 \to \mathbb{R}$. If g is differentiable at $c \in I_1$ and f is differentiable at $g(c) \in I_2$, then the chain rule says that

$$(f \circ g)'(c) = f'(g(c)) \cdot g'(c).$$

□

Note that

$$\begin{aligned}
\left(\frac{f}{g}\right)'(c) &= \lim_{x\to c} \frac{(f/g)(x) - (f/g)(c)}{x - c} \\
&= \lim_{x\to c} \frac{\frac{f(x)}{g(x)} - \frac{f(c)}{g(c)}}{x - c} \\
&= \lim_{x\to c} \frac{\frac{f(x)g(c)}{g(x)g(c)} - \frac{f(c)g(x)}{g(x)g(c)}}{x - c} \\
&= \lim_{x\to c} \left(\frac{1}{g(x)g(c)} \cdot \frac{f(x)g(c) - f(c)g(x)}{x - c}\right) \\
&= \lim_{x\to c} \frac{1}{g(x)g(c)} \cdot \lim_{x\to c} \frac{f(x)g(c) - f(c)g(x)}{x - c} && \text{(Limit law)}
\end{aligned}$$

For the first limit, we use the fact that f is differentiable and hence continuous. As for the second limit, we rewrite algebraically.

$$
\begin{aligned}
&= \frac{1}{g(c)g(c)} \cdot \lim_{x\to c} \frac{f(x)g(c) - f(c)g(c) + f(c)g(c) - f(c)g(x)}{x-c} \\
&= \frac{1}{g(c)g(c)} \cdot \left[\lim_{x\to c} \frac{f(x)g(c) - f(c)g(c)}{x-c} + \lim_{x\to c} \frac{f(c)g(c) - f(c)g(x)}{x-c}\right] \\
&= \frac{1}{g(c)g(c)} \cdot \left[\left(\lim_{x\to c} \frac{f(x)-f(c)}{x-c}\right) g(c) + f(c) \lim_{x\to c} \frac{g(c)-g(x)}{x-c}\right] \\
&= \frac{1}{g(c)g(c)} \cdot \left[\left(\lim_{x\to c} \frac{f(x)-f(c)}{x-c}\right) g(c) - f(c) \left(\lim_{x\to c} \frac{g(x)-g(c)}{x-c}\right)\right] \\
&= \frac{1}{g(c)g(c)} \cdot [f'(c)g(c) - f(c)g'(c)] \\
&= \frac{f'(c)g(c) - f(c)g'(c)}{g(c)^2}.
\end{aligned}
$$

□
□

Solution to Question 6. We have proved in class that

$$\frac{d}{dx}x^1 = 1 = 1x^{1-1}.$$

So the base case works. For the inductive step, assume that for some $k \in \mathbb{N}$ we have

$$\frac{d}{dx}x^k = kx^{k-1}.$$

Then, by the product rule and the above,

$$
\begin{aligned}
\frac{d}{dx}x^{k+1} &= \frac{d}{dx}(x \cdot x^k) \\
&= \left(\frac{d}{dx}x\right)(x^k) + (x)\left(\frac{d}{dx}x^k\right) \\
&= 1 \cdot x^k + x(kx^{k-1}) \\
&= x^k + kx^k \\
&= (k+1)x^k \\
&= (k+1)x^{(k+1)-1}.
\end{aligned}
$$

And so, by induction, it holds for all $n \in \mathbb{N}$. □
□

Solution to Question 7. Part (a) Let $g(x) = |f(x)|$. Since $f(c) \neq 0$, either $f(c) > 0$ or $f(c) < 0$.

First suppose $f(c) > 0$. Since f is differentiable at c, it is also continuous at c. So by Lemma 6.36, for all x sufficiently close to c, we also have $f(x) > 0$. Thus for all x sufficiently close to c,

$$|f(x)| = f(x) \qquad \text{and} \qquad |f(c)| = f(c).$$

Therefore,

$$
\begin{aligned}
g'(c) &= \lim_{x\to c} \frac{g(x)-g(c)}{x-c} \\
&= \lim_{x\to c} \frac{|f(x)| - |f(c)|}{x-c} \\
&= \lim_{x\to c} \frac{f(x)-f(c)}{x-c} \\
&= f'(c).
\end{aligned}
$$

Now suppose $f(c) < 0$. Again, since f is differentiable at c, it is continuous at c. So, again by Lemma 6.36, for all x sufficiently close to c, we also have $f(x) < 0$. Thus for all x sufficiently close to c,

$$|f(x)| = -f(x) \qquad \text{and} \qquad |f(c)| = -f(c).$$

Therefore,

$$\begin{aligned} g'(c) &= \lim_{x \to c} \frac{g(x) - g(c)}{x - c} \\ &= \lim_{x \to c} \frac{|f(x)| - |f(c)|}{x - c} \\ &= \lim_{x \to c} \frac{-f(x) - (-f(c))}{x - c} \\ &= \lim_{x \to c} \frac{-(f(x) - f(c))}{x - c} \\ &= -\lim_{x \to c} \frac{f(x) - f(c)}{x - c} \\ &= -f'(c). \end{aligned}$$

So in either case, $|f|$ is differentiable at c. In fact,

$$(|f|)'(c) = \begin{cases} f'(c), & \text{if } f(c) > 0, \\ -f'(c), & \text{if } f(c) < 0. \end{cases}$$

Part (b) Let $f(x) = x$ and let $c = 0$. Then f is differentiable at 0, and

$$f(0) = 0.$$

But

$$|f(x)| = |x|,$$

and as we discussed in class, $|x|$ is not differentiable at $x = 0$. Indeed,

$$\begin{aligned} (|f|)'(0) &= \lim_{x \to 0} \frac{|f(x)| - |f(0)|}{x - 0} \\ &= \lim_{x \to 0} \frac{|x| - 0}{x} \\ &= \lim_{x \to 0} \frac{|x|}{x}. \end{aligned}$$

If $x > 0$, then $\frac{|x|}{x} = 1$, while if $x < 0$, then $\frac{|x|}{x} = -1$. So if you approached from the right with $a_n = 1/n$, you will get a different answer than if you approached from the left with $b_n = -1/n$. Therefore, by Note 6.13, this limit does not exist. Thus $|f|$ is not differentiable at 0.

Solution to Question 8. Assume that f' is bounded. Then there exists some $M > 0$ such that $|f'(x)| \leq M$ for all $x \in I$.

Let $\epsilon > 0$. Let $\delta = \epsilon/M > 0$. Now, pick any $x, y \in I$ for which $|x - y| < \delta$. We aim to show that $|f(x) - f(y)| < \epsilon$. Since f is differentiable on I, by the Mean Value Theorem there exists some $c \in (x, y)$ such that

$$f'(c) = \frac{f(x) - f(y)}{x - y}.$$

Hence,

$$\begin{aligned}
|f'(c)| = \left|\frac{f(x)-f(y)}{x-y}\right| &\leq M \\
|f(x)-f(y)| &\leq M \cdot |x-y| \\
|f(x)-f(y)| &< M \cdot \delta \\
|f(x)-f(y)| &< M \cdot \frac{\epsilon}{M} \\
|f(x)-f(y)| &< \epsilon.
\end{aligned}$$

□
□

Solution to Question 9. From Question 2(b), we know that if $f(x) = \frac{1}{x}$, then

$$f'(x) = -\frac{1}{x^2}.$$

Now let $g : \mathbb{R} \to \mathbb{R}$ be differentiable, and consider the function

$$h(x) = \frac{1}{g(x)} = f(g(x)).$$

That is, h is the composition of $f(x) = \frac{1}{x}$ with $g(x)$.

Since g is differentiable and f is differentiable wherever $g(x) \neq 0$, we may apply the Chain Rule. Thus,

$$h'(x) = f'(g(x)) \cdot g'(x).$$

Substituting the formula for $f'(x)$, we obtain

$$h'(x) = -\frac{1}{(g(x))^2} \cdot g'(x).$$

Therefore,

$$\left(\frac{1}{g}\right)'(x) = -\frac{1}{g(x)^2} \cdot g'(x),$$

as desired.

□

Solution to Question 10. Part a. Consider the function $h(x) = f(x) - x$, and note that since f is differentiable and hence continuous on $[0, 3]$, so is h. Then $h(1) = 2 - 1 = 1$ and $h(3) = 2 - 3 = -1$, so by the intermediate value theorem there's some $c \in (1, 3)$ with $h(c) = 0$, meaning that $f(c) = c$. □

Part b. Since f is differentiable on $[0, 3]$, by the mean value theorem there is some $d \in [0, 3]$ such that $f'(d) = \frac{f(3)-f(0)}{3-0} = \frac{1}{3}$. □

Part c. First, since $f(1) = f(3)$ and f is differentiable, by Rolle's theorem there is a point $b \in (1, 3)$ with $f'(b) = 0$. And from part (b) above, there is a $d \in [0, 3]$ with $f'(d) = \frac{1}{3}$. Lastly, since $0 < \frac{1}{4} < \frac{1}{3}$, using Darboux's theorem there must be some e between b and d (and therefore in $[0, 3]$) satisfying $f'(e) = \frac{1}{4}$. □