Chapter 6 Solutions to Selected Exercises

Notes:

- The questions are in a separate PDF on LongFormMath.com.
- For most problems there are many correct solutions, so the below are not the only correct ways to solve the problems.
- If you spot an error, please email it to me at LongFormMath@gmail.com. Thanks!

Solution to Question 1.

Part (a). Scratch Work. Let $\epsilon > 0$. We start with the conclusion and then unwind it to get to a $|x-3|$ on the left, since this is the thing we can control.

$$\begin{aligned}
|f(x) - L| &< \epsilon \\
|2x + 5 - 11| &< \epsilon \\
|2x - 6| &< \epsilon \\
2 \cdot |x - 3| &< \epsilon \\
|x - 3| &< \frac{\epsilon}{2} \quad \checkmark \\
\text{So, set } \delta &= \frac{\epsilon}{2}.
\end{aligned}$$

Solution. Let $f(x) = 2x+5$ and $L = 11$. Let $\epsilon > 0$, and let $\delta = \frac{\epsilon}{2} > 0$. Then for any x for which $|x-3| < \delta$ we have

$$\begin{aligned}
|f(x) - L| &= |2x + 5 - 11| \\
&= |2x - 6| \\
&= 2 \cdot |x - 3| \\
&< 2 \cdot \delta \\
&= 2 \cdot \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

□

Part (b). Scratch Work. Let $\epsilon > 0$. We start with the conclusion and then unwind it to get to a $|x-0|$ on the left, since this is the thing we can control.

$$\begin{aligned}
|f(x) - L| &< \epsilon \\
|x^2 - 0| &< \epsilon \\
|x|^2 &< \epsilon \\
|x| &< \sqrt{\epsilon} \quad \checkmark \\
|x - 0| &< \sqrt{\epsilon} \\
\text{So, set } \delta &= \sqrt{\epsilon}.
\end{aligned}$$

Solution. Let $f(x) = x^2$ and $L = 0$. Let $\epsilon > 0$, and let $\delta = \sqrt{\epsilon} > 0$. Then for any x for which $|x - 0| < \delta$ we have

$$\begin{aligned} |f(x) - L| &= |x^2 - 0| \\ &= |x|^2 \\ &< (\delta)^2 \\ &< (\sqrt{\epsilon})^2 \\ &= \epsilon. \end{aligned}$$

□

Part (c). Scratch Work. Let $\epsilon > 0$. We start with the conclusion and then unwind it to get to a $|x - 1|$ on the left, since this is the thing we can control.

$$\begin{aligned} \left|\left(\frac{x^3}{x-1} - \frac{1}{x-1}\right) - 3\right| &< \epsilon \\ \left|\frac{x^3-1}{x-1} - 3\right| &< \epsilon \\ \left|\frac{(x-1)(x^2+x+1)}{x-1} - 3\right| &< \epsilon \\ \left|(x^2+x+1) - 3\right| &< \epsilon \\ \left|x^2+x-2\right| &< \epsilon \\ |(x+2)(x-1)| &< \epsilon \\ |(x+2)| \cdot |(x-1)| &< \epsilon \\ |(x-1)| &< \frac{\epsilon}{|x+2|} \quad \checkmark \end{aligned}$$

This one looks different than the others because the thing on the right has x's in it. Moreover, as x grows it becomes arbitrarily small, and so we can't say that it's always bigger than, say, $\epsilon/10$... right?

The trick is to remember that what we care about is that there is some small interval $(1-\delta, 1+\delta)$ around 1 which we get to choose in order to force the function to be within ϵ of 3. So since we can control δ, we can also control how big the x values can be. That is, we can make sure that $\frac{\epsilon}{|x+2|}$ is not arbitrarily small. For example, if we insist that $\delta \leq 1$, then $|x-1| < \delta$ implies $-1 < x - 1 < 1$, which implies $0 < x < 2$. Therefore we see that $|x + 2|$ can not be any bigger than 4. And so

$$\begin{aligned} |(x-1)| &< \frac{\epsilon}{|x+2|} \\ |(x-1)| &< \frac{\epsilon}{4} \\ \text{So, we want } \delta &\leq \frac{\epsilon}{4}, \text{ while also insisting } \delta \leq 1 \\ \text{That is, set } \delta &= \min\left\{1, \frac{\epsilon}{4}\right\} \end{aligned}$$

Solution. Let $f(x) = \frac{x^3}{x-1} - \frac{1}{x-1}$ and $L = 3$. Let $\epsilon > 0$, and let $\delta = \min\left\{1, \frac{\epsilon}{4}\right\} > 0$. Then for any x for which $|x - 1| < \delta$, first note that $|x - 1| < \min\left\{1, \frac{\epsilon}{4}\right\}$ implies that $|x - 1| < 1$, which implies $0 < x < 2$. This then means that $|x+2| < 4$. This fact will be used below. Now, for these x where $|x-1| < \delta$ we have

$$\begin{aligned}
|f(x) - L| &= \left|\frac{x^3}{x-1} - \frac{1}{x-1} - 3\right| \\
&= \left|\frac{x^3-1}{x-1} - 3\right| \\
&= \left|\frac{(x-1)(x^2+x+1)}{x-1} - 3\right| \\
&= \left|(x^2+x+1) - 3\right| \\
&= \left|x^2+x-2\right| \\
&= |(x+2)(x-1)| \\
&= |x+2| \cdot |x-1| \\
&\le 4 \cdot |x-1| && \text{We showed above that } |x+2| < 4 \\
&< 4 \cdot \delta \\
&\le 4 \cdot \frac{\epsilon}{4} \\
&= \epsilon.
\end{aligned}$$

□

Solution to Question 2. A function $f : A \to \mathbb{R}$ is *continuous* at a point $c \in A$ if for all $\epsilon > 0$ there exists $\delta > 0$ such that for all $x \in A$ where $|x - c| < \delta$ we have

$$|f(x) - f(c)| < \epsilon.$$

If f is continuous at every point in its domain, then f is called *continuous*.

Solution of Question 3.

Part (a). Scratch Work. Pick any $c \in \mathbb{R}$. We aim to prove that f is continuous at c. That is, for an $\epsilon > 0$ we want to find a $\delta > 0$ such that for all $x \in \mathbb{R}$ where $|x - c| < \delta$ we have

$$|f(x) - f(c)| < \epsilon.$$

Let's start with the conclusion and unravel it to determine which δ will give it.

$$\begin{aligned}
|f(x) - f(c)| &< \epsilon \\
|x^2 - c^2| &< \epsilon \\
|(x-c)(x+c)| &< \epsilon \\
|x-c| \cdot |x+c| &< \epsilon.
\end{aligned}$$

So we want to control the factor $|x+c|$. If we additionally require $|x - c| < 1$, then

$$\begin{aligned}
|x| &= |(x-c) + c| \\
&\le |x-c| + |c| \\
&< 1 + |c|,
\end{aligned}$$

and hence

$$\begin{aligned}
|x+c| &\le |x| + |c| \\
&< (1 + |c|) + |c| \\
&= 1 + 2|c|.
\end{aligned}$$

Therefore it would be enough to ensure

$$\begin{aligned}|x-c|(1+2|c|) &< \epsilon\\ |x-c| &< \frac{\epsilon}{1+2|c|}.\end{aligned}$$

So, let

$$\delta = \min\left\{1,\ \frac{\epsilon}{1+2|c|}\right\}.$$

Note that the first thing we did was choose a c. So c is fixed and so it's perfectly fine to let δ depend on c, just like it depends on ϵ.

Solution. Pick any $c \in \mathbb{R}$. We aim to prove that f is continuous at c. Let $\epsilon > 0$ and then let

$$\delta = \min\left\{1,\ \frac{\epsilon}{1+2|c|}\right\}.$$

Then for any $x \in \mathbb{R}$ for which $0 < |x-c| < \delta$ we have

$$\begin{aligned}|f(x)-f(c)| &= |x^2-c^2|\\ &= |(x-c)(x+c)|\\ &= |x-c|\cdot|x+c|.\end{aligned}$$

Since $|x-c| < 1$, we have

$$\begin{aligned}|x| &= |(x-c)+c|\\ &\le |x-c|+|c|\\ &< 1+|c|,\end{aligned}$$

and therefore

$$\begin{aligned}|x+c| &\le |x|+|c|\\ &< (1+|c|)+|c|\\ &= 1+2|c|.\end{aligned}$$

Thus,

$$\begin{aligned}|f(x)-f(c)| &< |x-c|(1+2|c|)\\ &< \delta(1+2|c|)\\ &\le \left(\frac{\epsilon}{1+2|c|}\right)(1+2|c|)\\ &= \epsilon.\end{aligned}$$

□

Part(b). Scratch Work. Pick any $c \in (0,\infty)$. We aim to prove that f is continuous at c. That is, for an $\epsilon > 0$ we want to find a $\delta > 0$ such that for all $x \in (0,\infty)$ where $|x-c| < \delta$ we have

$$|f(x)-f(c)| < \epsilon.$$

Let's start with the conclusion and unravel it to determine which δ will give it.

$$\begin{aligned}
|f(x)-f(c)| &< \epsilon \\
|\sqrt{x}-\sqrt{c}| &< \epsilon \\
\frac{|\sqrt{x}-\sqrt{c}|\cdot|\sqrt{x}+\sqrt{c}|}{|\sqrt{x}+\sqrt{c}|} &< \epsilon \\
\frac{|x-c|}{|\sqrt{x}+\sqrt{c}|} &< \epsilon \\
\frac{|x-c|}{\sqrt{x}+\sqrt{c}} &< \epsilon \\
|x-c| &< (\epsilon)(\sqrt{x}+\sqrt{c}) \\
|x-c| &< (\epsilon)(\sqrt{c}) \quad \checkmark \\
\text{So, let } \delta &= \epsilon\cdot\sqrt{c}
\end{aligned}$$

Note that the first thing we did was choose a c. So c is fixed and so it's perfect fine to let δ depend on c, just like it depends on ϵ.

Solution. Pick any $c \in (0,\infty)$. We aim to prove that f is continuous at c. Let $\epsilon > 0$ and then let $\delta = \epsilon\cdot\sqrt{c}$. Then for any $x \in (0,\infty)$ for which $0 < |x-c| < \delta$ we have

$$\begin{aligned}
|f(x)-f(c)| &= |\sqrt{x}-\sqrt{c}| \\
&= \frac{|\sqrt{x}-\sqrt{c}|\cdot|\sqrt{x}+\sqrt{c}|}{|\sqrt{x}+\sqrt{c}|} \\
&= \frac{|x-c|}{|\sqrt{x}+\sqrt{c}|} \\
&= \frac{|x-c|}{\sqrt{x}+\sqrt{c}} && \text{Since } \sqrt{x}+\sqrt{c}>0 \\
&< \frac{|x-c|}{\sqrt{c}} && \text{Since } \sqrt{x}>0 \\
&< \frac{\delta}{\sqrt{c}} \\
&= \frac{\epsilon\cdot\sqrt{c}}{\sqrt{c}} \\
&= \epsilon.
\end{aligned}$$

□

Solution of Question 4.

Part (a). False. Consider the following functions, both defined on $\mathbb{R}\to\mathbb{R}$.

$$f(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0. \end{cases} \qquad \text{and} \qquad g(x) = \begin{cases} -1 & \text{if } x \geq 0 \\ 1 & \text{if } x < 0. \end{cases}$$

And let $a = 0$. Then,

Then, neither function's limit exists, but $\lim_{x\to 0}[f(x)+g(x)] = \lim_{x\to 0} 0 = 0$.

Part (b). True. By the limit laws, if $\lim_{x\to a} f(x)$ exists, then $\lim_{x\to a} -f(x)$ exists, and if that limit exists and $\lim_{x\to a}[f(x)+g(x)]$ exists, then $\lim_{x\to a}\big(-f(x)+[f(x)+g(x)]\big) = \lim_{x\to a} g(x)$ exists. □

Solution to Question 5. Scratch Work. We will use the ϵ–δ definition of continuity. To prove that f is *discontinuous* at $c = 0$, it suffices to find an $\epsilon > 0$ such that *for every* $\delta > 0$ there exists an $x \in \mathbb{R}$ with $|x - 0| < \delta$ but

$$|f(x) - f(0)| \geq \epsilon.$$

First compute $f(0)$. Since $0 \geq 0$, we have

$$f(0) = 0 + 1 = 1.$$

Now, near 0 we can choose points on the negative side. If $x < 0$, then $f(x) = x - 1$, so

$$\begin{aligned}|f(x) - f(0)| &= |(x-1) - 1| \\ &= |x - 2|.\end{aligned}$$

If x is close to 0 and is negative, then $x - 1$ is close to -1, so the distance from $f(0) = 1$ should be close to 2, not close to 0. This suggests that we should choose an ϵ that is less than 2, such as $\epsilon = 1$.

Solution. We prove that f is discontinuous at $c = 0$. Note that

$$f(0) = 1.$$

Let $\epsilon = 1$. We will show that for every $\delta > 0$ there exists an $x \in \mathbb{R}$ such that $|x - 0| < \delta$ but $|f(x) - f(0)| \geq \epsilon$.

So pick any $\delta > 0$. Let

$$x = -\frac{\delta}{2}.$$

Then $x < 0$ and

$$|x - 0| = \left| -\frac{\delta}{2} \right| = \frac{\delta}{2} < \delta.$$

Since $x < 0$, we have $f(x) = x - 1$, and therefore

$$\begin{aligned}|f(x) - f(0)| &= |(x-1) - 1| \\ &= |x - 2| \\ &= \left| -\frac{\delta}{2} - 2 \right| \\ &= 2 + \frac{\delta}{2} \\ &\geq 2 \\ &> 1 \\ &= \epsilon.\end{aligned}$$

Thus, for this choice of ϵ, no matter how small δ is chosen, we can find an x with $|x| < \delta$ but $|f(x) - f(0)| \geq \epsilon$. Hence f is not continuous at 0, i.e. f is discontinuous at $c = 0$. □

Solution to Question 6.

— h is discontinuous at every rational —

First let $c \in \mathbb{Q}$. We aim to show that h is discontinuous at c. We will use Note 6.18, which says that if we can find a sequence a_n which converges to c but for which

$$\lim_{n \to \infty} h(a_n) \neq h(c),$$

then we may conclude that h is discontinuous at 0. Note that since c is rational, $h(c) > 0$.

Define $a_n = c + \dfrac{\sqrt{2}}{n}$. Note that

$$\lim_{n\to\infty} c + \frac{\sqrt{2}}{n} = c + 0 = c.$$

And so $a_n \to c$. We also claim that $a_n \notin \mathbb{Q}$ for any n. To show this, assume for a contradiction that $a_N = \frac{a}{b}$ for some integers N, a, and b. That is,

$$\begin{aligned}
\sqrt{2} + \frac{1}{N} &= \frac{a}{b}\\
\frac{N\sqrt{2}+1}{N} &= \frac{a}{b}\\
N\sqrt{2}+1 &= \frac{Na}{b}\\
\sqrt{2} &= \frac{1}{N}\left(\frac{Na}{b} - 1\right)\\
\sqrt{2} &= \frac{Na-b}{Nb}.
\end{aligned}$$

And since N, a and b are all integers, this implies that $\sqrt{2} \in \mathbb{Q}$, which we know to be a contradiction.

Therefore we see that $a_n \to c$ but $a_n \notin \mathbb{Q}$ for all n. Therefore $h(a_n) = 0$ for all n, and so

$$\lim_{n\to\infty} h(a_n) = 0 \neq f(c).$$

So by Note 6.18 we are done.

— h is continuous at every irrational —

Let $\epsilon > 0$. We will use the definition of continuity. That is, we wish to find a $\delta > 0$ so that if $0 < |x-c| < \delta$, then $|f(x) - f(c)| < \epsilon$.

By the Archimedean principle there exists some $N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon$. Note that there are only finitely many real numbers x for which $h(x) > \frac{1}{N}$ and also x has distance less than $1/2$ of x. To see this, just note that within distance $1/2$ there can be at most 1 integer, at most two of the form $\frac{m}{2}$, at most 3 of the form $\frac{m}{3}$, ..., at most $N-1$ of them form $\frac{m}{N-1}$. So in total, there are only finitely many.

Every other rational number has a larger denominator when written in lowest terms, and hence gets mapped to some $\frac{1}{k}$ where $k \geq N$. Therefore

$$Y = \left\{ y \in \mathbb{R} : h(y) > \frac{1}{N} \text{ and } |y - c| < \frac{1}{2} \right\}$$

is a finite set, implying that the set

$$\left\{ |c-y| \in \mathbb{R} : h(y) > \frac{1}{N} \text{ and } |y - c| < \frac{1}{2} \right\}$$

is also a finite set and hence its minimum exists. Notice that this is the set of distances between c and these terms, and that each of these distances is positive since $c \in\notin \mathbb{Q}$ while each $y \in \mathbb{Q}$. Let

$$\delta = \min\left\{ y \in \mathbb{R} : h(y) > \frac{1}{N} \right\}.$$

Then if $|x - c| < \delta$ then we may conclude that $x \notin Y$, and in general that $h(x) \not> \epsilon$, and hence

$$h(x) \leq \frac{1}{N} < \epsilon.$$

In summary, for any x for which $|x-c|<\delta$ we have

$$|h(x)-h(c)| = |h(x)-0| = h(x) < \frac{1}{N} < \epsilon,$$

concluding the proof. □

Solution to Question 7. Notice that, according to the extreme value theorem, if such a function were continuous, then this would be impossible. We should be looking for a discontinuous function. One example is the function $f:[0,1]\to\mathbb{R}$ where

$$f(x) = \begin{cases} x & \text{if } x < 0.5 \\ 0 & \text{if } x \geq 0.5. \end{cases}$$

□

Solution to Question 8. For part (a), it is possible. For instance, consider $f(x)=\sin(x)$ with $A=(0,100)$ and $B=[0,1]$. Then, notice that $f(A)=B$.

For part (b), it is not possible. Since A is a finite closed interval, it is both closed and bounded and hence compact. So by the Extreme Value Theorem, f attains a supremum and infimum on A. But if $f(A)=B$ where B is some open interval (a,b), then this is clearly not the case, as the supremum of this set (and hence of f) is b, which is not in the set; and likewise for the infimum. □

Solution to Question 9. Note that we are looking for a point x in which $f(x)=f(x+1/2)$, and in order for x and $(x+1/2)$ to be in $[0,1]$ we must have $x\in[0,1/2]$.

Consider the function $g:[0,1/2]\to\mathbb{R}$ given by $g(x)=f(x)-f(x+1/2)$. Since f is continuous, so is g. Note that if $f(0)=f(1/2)$ then we are done, so assume that $f(0)\neq f(1/2)$. Then either $f(0)>f(1/2)$ or $f(0)<f(1/2)$; without loss of generality, let's assume it's the former. And since $f(0)=f(1)$, this then also implies that $f(1)>f(1/2)$. Therefore:

$$g(0)=f(0)-f(1/2)>0 \qquad \text{and} \qquad g(1/2)=f(1/2)-f(1)<0.$$

So g is a continuous function on $[0,1/2]$, and $g(0)$ and $g(1/2)$ have different signs. Therefore by the Intermediate Value Theorem with $\alpha=0$ (or, just by Proposition 11.12) we know that there exists some $c\in(0,1/2)$ for which $g(c)=0$; i.e., that $f(c)-f(c+1/2)=0$; i.e., that $f(c)=f(c+1/2)$.

So by letting $x=c$ and $y=c+1/2$, we are done. □

Solution to Question 10. If f is continuous on $[a,b]$ and α is any number between $f(a)$ and $f(b)$, then there is some $c\in(a,b)$ for which $f(c)=\alpha$.

□

Solution to Question 11.
Part (a). If f is the person's elevation on their way up the mountain, and g is their elevation on the way down the mountain, then the hiker problem can be viewed in terms of these functions.

Part (b). Define $h:[a,b]\to\mathbb{R}$ by

$$h(x)=f(x)-g(x).$$

Since f and g are continuous on $[a,b]$, the function h is continuous on $[a,b]$. Moreover,

$$h(a)=f(a)-g(a)<0 \quad \text{and} \quad h(b)=f(b)-g(b)>0.$$

Thus $h(a)$ and $h(b)$ have different signs. By Proposition 6.37, there exists a point $c\in[a,b]$ such that

$$h(c)=0.$$

But $h(c)=0$ means $f(c)-g(c)=0$, i.e.

$$f(c)=g(c).$$

This proves that $f(c)=g(c)$ for some $c\in[a,b]$.

□