Chapter 5 Solutions to Selected Exercises

Notes:

- The questions are in a separate PDF on LongFormMath.com.
- For most problems there are many correct solutions, so the below are not the only correct ways to solve the problems.
- If you spot an error, please email it to me at LongFormMath@gmail.com. Thanks!

Solution to Question 1. Let $A \subseteq \mathbb{R}$. Then the following are equivalent.

(i) A is compact.

(ii) A is closed and bounded.

(iii) If (a_n) is a sequence of numbers in A, then there is a subsequence (a_{n_k}) that converges to a point in A.

Solution to Question 2. A set A is *compact* if every open cover of A contains a finite subcover of A.

Solution of Question 3.

(a) $\mathbb{Z}$ is closed, but is not open and not compact.

(b) $\{0.9, 0.99, 0.999, 0.9999, \dots\}$ is neither open, closed, nor compact.

(c) $\{0.9, 0.99, 0.999, 0.9999, \dots\} \cup \{1\}$ is closed and compact, but not open.

(d) $(0, 1) \cup [3, 4]$ is neither open, closed, nor compact.

(e) $[0, 1) \cup [1, 2]$ is closed and compact, but not open.

(f) $\mathbb{R} \setminus \mathbb{Q}$ is neither open, closed, nor compact.

(g) $\mathbb{R} \setminus \mathbb{Z}$ is open, but is not closed and not compact.

(h) $\{22\}$ is closed and compact, but not open.

Solution of Question 4.

(a) The set of limit points of $\mathbb{Z}$ is $\varnothing$.

(b) The set of limit points of $\{0.9, 0.99, 0.999, 0.9999, \dots\}$ is $\{1\}$.

(c) The set of limit points of $\{0.9, 0.99, 0.999, 0.9999, \dots\} \cup \{1\}$ is $\{1\}$.

(d) The set of limit points of $(0, 1) \cup [3, 4]$ is $[0, 1] \cup [3, 4]$.

(e) The set of limit points of $[0, 1) \cup [1, 2]$ is $[0, 2]$.

(f) The set of limit points of $\mathbb{R} \setminus \mathbb{Q}$ is $\mathbb{R}$.

(g) The set of limit points of $\mathbb{R} \setminus \mathbb{Z}$ is $\mathbb{R}$.

(h) The set of limit points of $\{22\}$ is $\varnothing$.

Solution to Question 5(a). One example is

$$[-1,0] \cup \bigcup_{n=1}^{\infty} \left(\frac{1}{n+1}, \frac{1}{n}\right],$$

which is a complicated way to write $[-1,1]$ which is closed and bounded, hence compact.

Solution to Question 6. There are many answers, but here is one. Consider the set of sets

$$\left\{\left(2, 8+\frac{1}{n}\right) : n \in \mathbb{N}\right\}.$$

For each n, the set $\left(2, 8+\frac{1}{n}\right)$ is an open interval and hence is open. However the intersection of all of these is

$$\bigcap_{n=1}^{\infty} \left(2, 8+\frac{1}{n}\right) = (2,8],$$

which is not open or closed.

Solution to Question 7. Part (a). Since the complement of a closed set is an open set, note that if $\{U_1, U_2, \dots, U_n\}$ is a collection of closed sets, then $\{{U_1}^c, {U_2}^c, \dots, {U_n}^c\}$ is a collection of open sets. Since the finite intersection of open sets is open, this then implies that

$$\bigcap_{k=1}^{n} {U_k}^c$$

is also open. And so by De Morgan's Laws,

$$\bigcap_{k=1}^{n} {U_k}^c = \left(\bigcup_{k=1}^{n} U_k\right)^c$$

is open. But if this is open, then we can conclude that its complement,

$$\bigcup_{k=1}^{n} U_k,$$

is closed, as desired.

Part (b). Likewise to part (a), note that if $\{U_\alpha\}$ is a collection of closed sets, then $\{{U_\alpha}^c\}$ is a collection of open sets. Since the arbitrary union of open sets is open, this then implies that

$$\bigcup_{\alpha} {U_\alpha}^c$$

is also open. And so by De Morgan's Laws,

$$\bigcup_{\alpha} {U_\alpha}^c = \left(\bigcap_{\alpha} U_\alpha\right)^c$$

is open. But if this is open, then we can conclude that its complement,

$$\bigcap_{\alpha} U_\alpha,$$

is closed, as desired. □

Solution to Question 8. First assume that A is compact, and hence closed and bounded. Given an arbitrary sequence (a_n) from A, we aim to show that this sequence has a subsequence which converges to a point in A. Because A is bounded and (a_n) is from A, we know that (a_n) is a bounded sequence. Applying the Bolzano-Weierstrass theorem, this then implies that (a_n) has a convergent subsequence. This subsequence is of course also from A; so we have a convergent sequence from the closed set A, and hence by Theorem 5.10 from the book, this sequence must converge to a point in A, as desired.

We will prove the reverse direction by contraposition. That is, we will show that if A is not compact, then A does not have the property that every sequence from it has a subsequence which converges to a point in A. Note that being not compact means that A is either not closed or bounded (or both). So this is our assumption. As for our conclusion, we which to show that it is not true that every sequence has a subsequence converging to some point in A. Thus it suffices to construct a sequence which does not have a subsequence converging to a point in A.

Now that we have established what we have and what we need to show, assume that A is either not closed or not bounded.

<u>Case 1: A is not closed.</u> Since A is not closed, by Theorem 5.10 from the book there exists a sequence (a_n) from A which is converging to some x such that $x \notin A$. By Proposition 3.31 from the book, every subsequence of (a_n) must also converge to this same $x \notin A$. Therefore no subsequence converges to a point in A, completing this case.

<u>Case 2: A is not bounded.</u> Since A is not bounded, for each $n \in \mathbb{N}$ there exists a point $a_n \in A$ for which $|a_n| > n$. This is an infinite sequence of points, so either the subsequence of positive points has infinitely many terms, or the subsequence of negative points has infinitely many terms. Without loss of generality, assume it's the positive case. That is, (a_{n_k}) is itself a sequence of points from A which diverges to ∞. And hence every subsequence of (a_{n_k}) also diverges to ∞. That is, no subsequence is converging to a point in A, as desired. □

□

Solution to Question 9. Answers vary. Here is one:

$$\left\{ \left(0, 7 - \frac{1}{11}\right), (6.9, 7.2) \right\}.$$

□

Solution to Question 10.

(a) Intuitively, a set $A \subseteq \mathbb{R}$ is *connected* if it is "all in one piece," meaning you cannot split A into two nonempty parts that are separated from each other by open sets (equivalently, there is no "gap" in A that cleanly divides it into two separated chunks).

(b) One example is the set $(3, 4) \cup (4, 5)$.

(c) One example are the sets $A = [2, 3] \cup [4, 5]$, and $B = (3, 4)$.

(d) No, $\mathbb{Z}$ is not connected, which can be seen in the definition of connectedness by letting

$$U = (-\infty, 3.5) \qquad \text{and} \qquad V = (3.5, \infty).$$

Then U and V are open, $U \cap V = \emptyset$, and

$$U \cap \mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, 3\} \neq \emptyset, \qquad V \cap \mathbb{Z} = \{4, 5, 6, 7, \dots\} \neq \emptyset.$$

Moreover,

$$(U \cap \mathbb{Z}) \cup (V \cap \mathbb{Z}) = \{\dots, -2, -1, 0, 1, 2, 3\} \cup \{4, 5, 6, 7, \dots\} = \mathbb{Z}.$$

Thus $\mathbb{Z}$ is not connected.

(e) No, $\mathbb{Q}$ is not connected, which can be seen in the definition of connectedness by letting

$$U = (-\infty, \sqrt{2}) \qquad \text{and} \qquad V = (\sqrt{2}, \infty).$$

Then U and V are open and disjoint. And every rational is either less than $\sqrt{2}$ or greater than $\sqrt{2}$, so

$$(U \cap \mathbb{Q}) \cup (V \cap \mathbb{Q}) = \mathbb{Q}.$$

Hence $\mathbb{Q}$ is not connected.

(f) Assume A is an interval, and suppose for contradiction that A is not connected. Then there exist disjoint open sets U, V such that $U \cap A$ and $V \cap A$ are both nonempty and

$$(U \cap A) \cup (V \cap A) = A.$$

Choose $u \in U \cap A$ and $v \in V \cap A$, and assume $u < v$ (otherwise swap them). Since A is an interval, every x with $u < x < v$ lies in A.

Let

$$S = \{x \in A \cap U : x < v\}.$$

Then S is nonempty (since $u \in S$) and bounded above by v, so $\alpha = \sup S$ exists in $\mathbb{R}$. We claim $\alpha \in A$ (because A is an interval containing points below v and also v, so the whole segment between those points lies in A, and in particular α lies between elements of A).

Now $\alpha \notin U$: if $\alpha \in U$, since U is open there is $\epsilon > 0$ with $(\alpha - \epsilon, \alpha + \epsilon) \subset U$, and because $\alpha = \sup S$ there exists $x \in S$ with $\alpha - \epsilon/2 < x \leq \alpha$, hence $x + \epsilon/2 \in A \cap U$ and $x + \epsilon/2 < v$, contradicting that α is an upper bound of S. Thus $\alpha \notin U$.

Similarly, $\alpha \notin V$: if $\alpha \in V$, since V is open there is $\epsilon > 0$ with $(\alpha - \epsilon, \alpha + \epsilon) \subset V$. By definition of supremum there exists $x \in S$ with $\alpha - \epsilon/2 < x \leq \alpha$, but then $x \in U$ and also $x \in (\alpha - \epsilon, \alpha + \epsilon) \subset V$, contradicting $U \cap V = \emptyset$.

So $\alpha \notin U$ and $\alpha \notin V$, hence $\alpha \notin U \cup V$. But $\alpha \in A$ and $A \subset (U \cap A) \cup (V \cap A) \subset U \cup V$, a contradiction. Therefore no such U, V exist and A is connected. □

(g) Let $A \subseteq \mathbb{R}$ have more than one element.

(⇒) Assume A is connected. We prove A is an interval by showing that, given any two elements in A, call them a and b, that it is the case that every number between a and b is also in A.

To this end, consider any $a, b \in A$ with $a < b$. We claim that $(a, b) \subseteq A$. Suppose not; then there exists $c \in (a, b)$ with $c \notin A$. Consider the open sets

$$U = (-\infty, c) \qquad \text{and} \qquad V = (c, \infty).$$

They are disjoint and open. Moreover, since $c \notin A$ and every other real number is either less than c or greater than c, every point of A must be either in U or in V, so

$$(U \cap A) \cup (V \cap A) = A.$$

This contradicts connectedness. Therefore $(a, b) \subseteq A$, so A is an interval.

(⇐) This was done in part (f). □