Chapter 4 Solutions to Selected Exercises

Notes:

- The questions are in a separate PDF on LongFormMath.com.
- For most problems there are many correct solutions, so the below are not the only correct ways to solve the problems.
- If you spot an error, please email it to me at LongFormMath@gmail.com. Thanks!

Solution to Question 1. Part (a) converges by the alternating series test. Part (b) diverges by the k^{th}-term test. Part (c) converges by the geometric series test. □

□

Solution to Question 2. We will prove it via the comparison test. Note that $-|a_k| \leq a_k \leq |a_k|$ for all k, implying that

$$0 \leq a_k + |a_k| \leq 2|a_k|$$

for all k. We are given that $\sum_{k=1}^{\infty} |a_k|$ converges, which by the series limit laws means that $\sum_{k=1}^{\infty} 2|a_k|$ converges. Combining this with the inequality above, by the comparison test we deduce that $\sum_{k=1}^{\infty}(a_k + |a_k|)$ converges too.

From here, the series limit laws will finish it off for us. Since $\sum_{k=1}^{\infty} |a_k|$ converges, also $\sum_{k=1}^{\infty} -|a_k|$ converges. We have at this point shown that

$$\sum_{k=1}^{\infty}(a_k + |a_k|) \qquad \text{and} \qquad \sum_{k=1}^{\infty} -|a_k|$$

both converge. Applying the limit laws one last time, this means that their sum

$$\sum_{k=1}^{\infty}(a_k + |a_k| - |a_k|) \;=\; \sum_{k=1}^{\infty} a_k$$

also converges, completing the proof. □

Solution of Question 3.

Part (a). Answers will vary, but one example is the series where $a_k = 1/k$.

Part (b). Since $\sum_{k=1}^{\infty} a_k$ converges, by the k^{th}-term test we have $a_k \to 0$. By the definition of sequence converges (with $\epsilon = 1$), there exists some N such that $|a_k - 0| < 1$ for all $k > N$. Combined with the assumption that $0 < a_k$, this means that $0 < a_k < 1$ for all $n > N$. Since there are at most N terms that are at least 1, by Note 4.12 in the book we may change all terms which are larger than 1 to be 1/2, and doing so does not change the convergence of the series. Call this new series

$$\sum_{k=1}^{\infty} \tilde{a}_k.$$

What we get is a convergent series where every term has $0 < \tilde{a}_k < 1$. And so $0 < \tilde{a}_k^2 < \tilde{a}_k$. And since $\displaystyle\sum_{k=1}^{\infty} \tilde{a}_k$ converges, by the comparison test we see that $\displaystyle\sum_{k=1}^{\infty} \tilde{a}_k^2$ also converges.

And since $\displaystyle\sum_{k=1}^{\infty} \tilde{a}_k^2$ and $\displaystyle\sum_{k=1}^{\infty} a_k^2$ also differ in only finitely many terms, the latter must converge too, as desired. □

Part (c). First observe that $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{\sqrt{k}}$ converges by the alternating series test, because $(\frac{1}{\sqrt{k}})$ is monotonically decreasing and $\frac{1}{\sqrt{k}} \to 0$. Next observe that

$$\sum_{k=1}^{\infty}\left((-1)^{k+1}\frac{1}{\sqrt{k}}\right)^2 = \sum_{k=1}^{\infty}\frac{1}{k},$$

which is the harmonic series, which we showed in class (Proposition 4.14 in the book) diverges. □
□

Solution of Question 4. Answers vary. One example for part (a) is where $a_k = (-1)^k \frac{1}{k}$. And one example for part (b) is there series $1 - 1 + 1 - 1 + 1 - 1 + \dots$, whose sequence of partial sums is $1, 0, 1, 0, 1, 0, 1, \dots$, which has the convergent subsequence $1, 1, 1, 1, \dots$. □

Solution to Question 5.
Part (a). Note that

$$77.777777\ldots = 70 + 7 + 0.7 + 0.07 + 0.007 + \cdots = \sum_{k=0}^{\infty}\frac{70}{10^k}.$$

This geometric series has $a = 70$ and ratio $r = \dfrac{1}{10}$. Thus, by the geometric series formula,

$$77.777777\cdots = \frac{70}{1-\frac{1}{10}} = \frac{70}{9/10} = \frac{700}{9}.$$

This is a ratio of integers, so the number is rational.

Part (b). Another decomposition is found by noting

$$77.777777\ldots = 77 + 0.77 + 0.0077 + 0.000077 + \cdots = \sum_{k=0}^{\infty}\frac{77}{100^k}.$$

This geometric series has $a = 77$ and ratio $r = \dfrac{1}{100}$. Thus, by the geometric series formula,

$$77.777777\cdots = \frac{77}{1-\frac{1}{100}} = \frac{77}{99/100} = \frac{7700}{99}.$$

This is a ratio of integers, so the number is rational. In fact, it is the same fraction from part (a), since 7700 and 99 can both be divided by 11 to obtain $\frac{700}{9}$.

Part (c). Suppose the decimal expansion of q eventually repeats. Then there exist nonnegative integers n (preperiod length) and $m \geq 1$ (period length), an integer A, and digits forming one repeating block represented by an integer B with $0 \leq B \leq 10^m - 1$, such that

$$q = \frac{A}{10^n} + \frac{B}{10^{n+m}} + \frac{B}{10^{n+2m}} + \frac{B}{10^{n+3m}} + \cdots = \frac{A}{10^n} + \frac{B}{10^n}\left(\frac{1}{10^m} + \frac{1}{10^{2m}} + \frac{1}{10^{3m}} + \cdots\right).$$

Here, $\frac{A}{10^n}$ encodes the finite part up to the start of repetition, and $\frac{B}{10^n}$ encodes one period shifted to begin at the first repeating digit.

The infinite sum in parentheses is geometric with first term $a = \frac{1}{10^m}$ and common ratio $r = \frac{1}{10^m}$, so

$$\left(\frac{1}{10^m} + \frac{1}{10^{2m}} + \frac{1}{10^{3m}} + \cdots\right) = \frac{\frac{1}{10^m}}{1 - \frac{1}{10^m}} = \frac{1}{10^m - 1}.$$

Thus

$$q \;=\; \frac{A}{10^n} + \frac{B}{10^n} \cdot \frac{1}{10^m - 1} \;=\; \frac{A(10^m-1)+B}{10^n(10^m-1)}.$$

The right-hand side is a ratio of integers, hence q is rational.

$\square$

Solution to Question 6. Let $\sum_{k=1}^{\infty} a_k$ be a series with $a_k \neq 0$ and set

$$r = \lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right|.$$

Assume $r < 1$. We prove that $\sum_{k=1}^{\infty} |a_k|$ converges.

(a) Let q satisfy $r < q < 1$. Then $\varepsilon = q - r > 0$. By the definition of limit, there exists $N \in \mathbb{N}$ such that for all $k > N$, we have

$$\left|\left|\frac{a_{k+1}}{a_k}\right| - r\right| < \varepsilon$$

Thus, for $k > N$,

$$\left|\frac{a_{k+1}}{a_k}\right| < r + \varepsilon = q.$$

Multiplying by $|a_k|$ gives, for all $k \geq N$,

$$|a_{k+1}| \leq q\,|a_k|.$$

(b) Because $0 < q < 1$, the geometric series $\sum_{m=1}^{\infty} |a_N|\, q^m$ converges. Indeed,

$$\sum_{m=1}^{\infty} |a_N|\, q^m = |a_N| \cdot \frac{q}{1-q}.$$

(c) First, from part (a) we showed that $|a_{k+1}| \leq q\,|a_k|$ for all $k > N$. Thus,

$$|a_{N+2}| \leq q|a_{N+1}|,$$
$$|a_{N+3}| \leq q|a_{N+2}| \leq q^2|a_{N+1}|,$$
$$\ldots$$
$$|a_{N+m}| \leq q^{m-1}|a_{N+1}|, \quad \text{for any } m \in \mathbb{N}.$$

Hence, for the tail of $\sum |a_k|$,

$$\sum_{k=N+1}^{\infty} |a_k| \;\leq\; \sum_{m=0}^{\infty} |a_{N+1}|\, q^m \;=\; |a_{N+1}| \cdot \frac{1}{1-q}.$$

Since a finite sum $\sum_{k=1}^{N} |a_k|$ must equal a finite number L, adding it to a convergent tail shows

$$\sum_{k=1}^{\infty} |a_k| = L + |a_{N+1}| \cdot \frac{1}{1-q},$$

which is the sum of two finite numbers, hence is finite. Therefore, we have shown that $\sum a_k$ converges absolutely. $\square$

Solution to Question 7. Recall that to show a sequence (b_n) diverges to ∞ we need to show that for any $M > 0$ there exists a point N such that $b_n > M$ for all $n > N$. So once we construct our rearrangement we will prove that it satisfies this definition.

To this end, let p_k be the k^{th} positive (or zero) term of (a_n) and let n_k be the k^{th} negative term of (a_n). Since $\sum_{k=1}^{\infty} a_k$ converges conditionally, we must have:

$$\sum_{k=1}^{\infty} p_k = \infty \qquad \text{and} \qquad \sum_{k=1}^{\infty} n_k = -\infty.$$

Since $\sum_{k=1}^{\infty} p_k = \infty$ there exists some P_1 such that

$$\sum_{k=1}^{P_1} p_k > 1 + (-n_1) > 1.$$

Likewise, there must exist some $P_2 > P_1$ such that

$$\left(\sum_{k=1}^{P_1} p_k\right) + n_1 + \left(\sum_{k=P_1+1}^{P_2} p_k\right) > 2 + (-n_2) > 2.$$

Likewise there must exist some $P_3 > P_2$ such that

$$\left(\sum_{k=1}^{P_1} p_k\right) + n_1 + \left(\sum_{k=P_1+1}^{P_2} p_k\right) + n_2 + \left(\sum_{k=P_2+1}^{P_3} p_k\right) > 3 + (-n_3) > 3.$$

Continuing in this way, we obtain a sum of all the p_i's and n_i's (meaning it is indeed a rearrangement) such that for any $M > 0$, we know that the partial rearrangement

$$\left(\sum_{k=1}^{P_1} p_k\right) + n_1 + \cdots + \left(\sum_{k=P_{M-1}+1}^{P_M} p_k\right) > M + (-n_M) > M.$$

Moreover, we now claim that after this point in our rearrangement the sum will never again be less than M. The reason is that the next term in the sum is n_M, but since by the inequality our sum is currently larger than $M + (-n_m)$, adding this will not bring the sum below M. Furthermore, next we will add enough positive terms so that after the next negative term (n_{M+1}) we will remain above $M + 1$. And then we will add enough positive terms so that after the next negative term (n_{M+2}) we will remain above $M + 2$. This will continue forever; so indeed, we will always remain above M. □

Solution to Question 8. Let $M > 0$. Since $\sum_{k=1}^{\infty} a_k = \infty$ there exists some N such that

$$\sum_{k=1}^{n} a_k > M$$

for all $n > N$ (in particular, $n = N + 1$ satisfies the above). Consider an arbitrary rearrangement $\sum_{k=1}^{\infty} b_k$ of $\sum_{k=1}^{\infty} a_k$, and let $f : \mathbb{N} \to \mathbb{N}$ be a bijection giving it; that is, $b_{f(k)} = a_k$. Let

$$N' = \max\{f(k) : k \in \{1, 2, 3, \ldots, N + 1\}\}.$$

Intuitively, the first $N+1$ terms of $\sum_{k=1}^{\infty} a_k$ were permuted around by the bijection f and now are in different positions in the sum $\sum_{k=1}^{\infty} b_k$; perhaps the first term $\sum_{k=1}^{\infty} a_k$ is now the $f(1) = 32^{\text{nd}}$ term in $\sum_{k=1}^{\infty} b_k$, and the second term in $\sum_{k=1}^{\infty} a_k$ is now the $f(2) = 8^{\text{th}}$ term in $\sum_{k=1}^{\infty} b_k$, and so on. So we look at where all of the first $N+1$ terms from $\sum_{k=1}^{\infty} a_k$ went, and we let N' be the last location of these terms in the rearrangement $\sum_{k=1}^{\infty} b_k$.

From this we see that $b_1, b_2, b_3, \dots, b_{N'}$ contains all of the terms from $a_1, a_2, a_3, \dots, a_N$ (and probably many more). And since by assumption all the terms are non-negative, this in particular means that

$$\sum_{k=1}^{N'} b_k \;\geq\; \sum_{k=1}^{N} a_k.$$

And again, since all the terms are non-negative, we have that $\displaystyle\sum_{k=1}^{n} b_k \geq \sum_{k=1}^{N'} b_k$ for any $n > N'$. Putting this all together,

$$\sum_{k=1}^{n} b_k \;\geq\; \sum_{k=1}^{N'} b_k \;\geq\; \sum_{k=1}^{N} a_k > M$$

for all $n > N'$. So by definition, $\sum_{k=1}^{\infty} b_k = \infty$.

We have shown that if $\sum_{k=1}^{\infty} a_k = \infty$ then an arbitrary rearrangement of this series also diverges to ∞. This completes the proof. □

Solution to Question 9. Answers vary.

Solution to Question 10. Answers vary.