

Chapter 8 Solved Exercises

Question 1. (Similar to Exercise 8.1) Use Newton's method of fluxions to take the following derivatives.

(a) Suppose $y = 4x + 3$. Show that $\frac{\dot{y}}{\dot{x}} = 4$.

(b) Suppose $y = 2x^2 + 3x$. Show that $\frac{\dot{y}}{\dot{x}} = 4x + 3$.

(c) Suppose $y = 8$. Show that $\frac{\dot{y}}{\dot{x}} = 0$.

Question 2. (Similar to Exercise 8.2(a)) Suppose $y = 2\sqrt{x}$. Use Newton's method of fluxions to show that $\frac{\dot{y}}{\dot{x}} = \frac{1}{\sqrt{x}}$.

Question 3. (Similar to Exercise 8.3) Suppose $x^2 + 5xy - y^2 = 0$. Use Newton's method of fluxions to find $\frac{\dot{y}}{\dot{x}}$.

Question 4. (Exercise 8.5) Prove the quotient rule. That is, suppose u and v are fluents which depend on a variable t , and $w = \frac{u}{v}$. Use Newton's method of fluxions to prove that

$$\dot{w} = \frac{v\dot{u} - u\dot{v}}{v^2}.$$

Hint: One way to proceed is to note that $w = \frac{u}{v}$ implies that $u = vw$, and this lets you use the product rule.

Question 5. (Exercise 8.10) An important representation of π is derived from a series traditionally called the *Gregory-Leibniz series* — although it was first discovered by Madhava of Sangamagrama. It is this:

$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots .$$

In this problem, you will derive this series using methods available at the time, and then find the formula for π .

- (a) Use the inverse trigonometric substitution $t = \tan(\theta)$ to find this integral representation of $\arctan(x)$:

$$\int_0^x \frac{1}{1+t^2} dt = \arctan(x).$$

Note: You cannot assume that $\frac{d}{dx} \arctan x = \frac{1}{1+x^2}$; in your calculus textbook this was probably proved using implicit differentiation, which was developed much later.

- (b) Expand $\frac{1}{1+t^2}$ using either the geometric series theorem (Theorem ??) or the Binomial theorem. Then, integrate term-by-term from $t = 0$ to $t = x$ to obtain the power series for $\arctan(x)$. Note where the series converges.

- (c) Set $x = 1$ in your series to show that:

$$\pi = 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \frac{4}{9} - \frac{4}{11} + \dots$$

- (d) Using this expression, approximate π using the first six terms. Is this a good approximation? What do you notice about the rate of convergence?

Question 6. (Similar to Exercise 8.11) Let

$$H_n = \sum_{k=1}^n \frac{1}{k}$$

be the n^{th} harmonic number. There are many proofs that the Harmonic series diverges. In this exercise, provide a proof of this fact by arguing that

$$H_9 > \frac{9}{10} \quad \text{and} \quad H_{99} > 2 \cdot \frac{9}{10} \quad \text{and} \quad H_{999} > 3 \cdot \frac{9}{10},$$

and in general that

$$H_{10^k-1} > k \cdot \frac{9}{10},$$

and explaining why this proves the theorem.

Question 7. (Similar to Exercise 8.11) My favorite proof that the harmonic series diverges is a proof-by-contradiction published by Arthur Cusumano in 1998. The proof relies on this observation. Fill in the details to complete the proof.

$$\begin{aligned} S &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \dots \\ &= \left(1 + \frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6}\right) + \left(\frac{1}{7} + \frac{1}{8}\right) + \dots \\ &= \left(1 + \frac{1}{2}\right) + \left(\frac{1}{2} + \frac{1}{12}\right) + \left(\frac{1}{3} + \frac{1}{30}\right) + \left(\frac{1}{4} + \frac{1}{56}\right) + \dots \end{aligned}$$

Question 8. (Exercise 8.17) Once the calculus was developed, mathematicians were off to the races to find interesting applications of this new theory. Some of this progress is discussed in Appendix A on topics in applied math. Read this appendix and write a 500-word essay about what you learned and found interesting. Also, pick a topic from that appendix, research it on your own and share what else you learned.

Question 9. (Exercise 8.21) Write a 1000-word essay on the lives and math accomplishments of the Bernoulli family.

Question 10. (Exercise 8.24) Come up with your own exercise that uses the information from this chapter. Make sure it is different from the other exercises in this chapter and have its level of difficulty be at about the same level as the others. Then, answer your question.