Chapter 5 Solved Exercises

- You must prove all of your answers (unless stated otherwise).
- Remember the proof techniques that you learned in your intro-to-proofs course: Direct, by contradiction, by using the contrapositive, by induction, and/or by cases. And you can disprove something by exhibiting a counterexample.

Question 1. State the *Heine-Borel, expanded theorem.*

Question 2. Define what it means for a set to be *compact.*

Question 3. (Similar to Exercise 5.1) For each of the following, determine whether the set is open, whether it is closed, and whether it is compact. (It might be more than one, or none of these.) You do not need to prove your answers.

(a) $\mathbb{Z}$

(b) $\{0.9, 0.99, 0.999, 0.9999, \ldots\}$

(c) $\{0.9, 0.99, 0.999, 0.9999, \ldots\} \cup \{1\}$

(d) $(0, 1) \cup [3, 4]$

(e) $[0, 1) \cup [1, 2]$

(f) $\mathbb{R} \setminus \mathbb{Q}$

(g) $\mathbb{R} \setminus \mathbb{Z}$

(h) $\{22\}$

Question 4. (Similar to Exercise 5.2) Determine the set of limit points for each of the eight sets from the previous question.

Question 5. (Similar to Exercise 5.5)

(a) Does there exist a compact set which is the union of infinitely many disjoint intervals? Give an example or prove that this is impossible.

(b) Make up your own "does there exist a. . . " question related to open sets, closed sets or compact sets. Then, answer your question.

Question 6. (Similar to Exercise 5.6(a)) Give an example of an infinite collection of open sets whose intersection is *not* open or closed. You should state which sets you choose and what their intersection is, but you do not need to prove your answer.

Question 7. (Exercise 5.11) Prove Proposition 5.12. That is, prove the following.

(a) If $\{U_1, U_2, \ldots, U_n\}$ is a collection of closed sets, then $\displaystyle\bigcup_{k=1}^{n} U_k$ is also a closed set.

(b) If $\{U_\alpha\}$ is a collection of closed sets, then $\bigcap_\alpha U_\alpha$ is also a closed set.

Question 8. (Exercise 5.17) Let $A \subseteq \mathbb{R}$. Prove that A is closed and bounded (i.e., compact) if and only if every sequence of numbers from A has a subsequence that converges to a point in A.

Question 9. (Similar to Exercise 5.23) One open cover of the set $[1,3] \cup [5,7]$ is the collection

$$\left\{ \bigcup_{n=1}^{\infty} \left(0, 7 - \frac{1}{n}\right) \right\} \cup \{(6.9, 7.2)\}.$$

Since $[1,3] \cup [5,7]$ is compact, we know that this open cover must have a finite subcover. Give an example of such a subcover of this cover. You do *not* need to prove your answer.

Definition. A set A of real numbers is said to be *connected* if there do not exist two open sets U and V such that

(i) $U \cap V = \emptyset$,

(ii) $U \cap A \neq \emptyset$ and $V \cap A \neq \emptyset$, and

(iii) $(U \cap A) \cup (V \cap A) = A$.

Question 10. (Similar to Exercise 5.33)

(a) Explain intuitively what it means for a set A to be connected, and give a couple examples of each.

(b) Give an example of a set A which is not connected, but $A \cup \{4\}$ is connected. You do not need to prove your answers.

(c) Give an example of a set A which is not connected, but there exists some open set B such that $A \cup B$ is connected.

(d) Is $\mathbb{Z}$ connected? Prove your answer.

(e) Is $\mathbb{Q}$ connected? Prove your answer.

(f) Prove that every interval is connected.

(g) Prove that if a set of real numbers has more than one element and is *not* an interval, then it is *not* connected.