Chapter 4 Solved Exercises

- You must prove all of your answers (unless stated otherwise).
- Remember the proof techniques that you learned in your intro-to-proofs course: Direct, by contradiction, by using the contrapositive, by induction, and/or by cases. And you can disprove something by exhibiting a counterexample.

Question 1. (Similar to Exercise 4.1) Determine whether each of the following converges conditionally, converges absolutely, or diverges. You do not need to prove your answers, but state which of the following tests gives the answers: the k^{th}-term test, the geometric series test, or the alternating series test.

(a) $\displaystyle\sum_{k=1}^{\infty}(-1)^k\frac{7\sqrt{k}}{k+2}$ (a) $\displaystyle\sum_{k=1}^{\infty}(-1)^k\frac{k^2}{k^2+7}$ (a) $\displaystyle\sum_{k=1}^{\infty}\frac{1}{e^k}$

Question 2. (Exercise 4.3) Prove Proposition 4.18. That is, prove that if $\displaystyle\sum_{k=1}^{\infty}|a_k|$ converges, then $\displaystyle\sum_{k=1}^{\infty}a_k$ converges too.

Question 3. (Exercise 4.4(a) and (b))

(a) Give an example of a series with nonnegative terms where $\displaystyle\sum_{k=1}^{\infty}a_k$ diverges, but $\displaystyle\sum_{k=1}^{\infty}a_k^2$ converges.

(b) Prove that if $\displaystyle\sum_{k=1}^{\infty}a_k$ converges where each $a_k>0$, then $\displaystyle\sum_{k=1}^{\infty}a_k^2$ converges.

Question 4. (Similar to Exercise 4.4)

(a) Give an example of a series where $\displaystyle\sum_{k=1}^{\infty}a_k$ converges, but $\displaystyle\sum_{k=1}^{\infty}(-1)^k a_k$ diverges.

(b) Give an example of a series that diverges, but its sequence of partial sums has a convergent subsequence.

Question 5. (Exercise 4.11)

(a) Find a way to write $77.77777777\ldots$ as a geometric series, and then prove this number is rational by using the geometric series test to write this number as a fraction with integers in the numerator and denominator.

(b) Write $77.77777777\ldots$ as a different geometric series, and use the geometric series test to write this number as a fraction with integers in the numerator and denominator. Are your two fractions the same?

(c) A number q has a *repeating decimal* if the non-integer portion of its decimal expansion is repetitive. For example, $72.578578578578578\ldots$ has a repeating decimal. Prove that if a number q has a repeating decimal, then q is rational.

Question 6. (Exercise 4.17) Prove the *ratio test* via the following steps. Given a series $\displaystyle\sum_{k=1}^{\infty} a_k$ with $a_k \neq 0$, let

$$r = \lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right|,$$

and assume that $r < 1$. We will prove that the series converges absolutely.

(a) Let q be such that $r < q < 1$. Explain why there is some N such that $n \geq N$ implies that $|a_{k+1}| \leq |a_k| \cdot q$.

(b) Explain why $\displaystyle\sum_{k=1}^{\infty} |a_N| \cdot q^k$ necessarily converges.

(c) Finally, use part (b) to prove that $\displaystyle\sum_{k=1}^{\infty} |a_k|$ converges.

Question 7. (Exercise 4.23) Show that if $\sum_{k=1}^{\infty} a_k$ is conditionally convergent, then there exists a rearrangement of this sum which diverges to ∞.

Question 8. (Exercise 4.27) Prove that if each $a_k \geq 0$ and $\sum_{k=1}^{\infty} a_k = \infty$, then any rearrangement of this sum also diverges to ∞.

Question 9. One of the most important series in math is called the *The Riemann zeta function*. Look up what this series is about and write a few paragraphs describing it and its importance.

Question 10. Using the Internet or another source, find another unsolved problem related to series. This could be a series that nobody knows what is sums to; such problems often ask whether the series has some property, like whether it is irrational or transcendental. Or it can be any other open problem related to series.