Chapter 3 Solutions to Selected Exercises

Notes:

- The questions are in a separate PDF on LongFormMath.com.
- For most problems there are many correct solutions, so the below are not the only correct ways to solve the problems.
- If you spot an error, please email it to me at LongFormMath@gmail.com. Thanks!

Solution to Question 1. A sequence (a_n) *converges* to $a \in \mathbb{R}$ if for all $\epsilon > 0$ there exists some N such that $|a_n - a| < \epsilon$ for all $n > N$.

Solution to Question 2. There are many answers, but here are a few.

(a). Let $a_n = (-1)^n$ and $a = 1$. Then for any $\epsilon > 0$, and any $N \in \mathbb{N}$, let $n = 2N$ (or any other even number larger than N). Then $|a_n - a| = |(-1)^{2N} - 1| = 0 < \epsilon$. So (a_n) does *Non*verge-type-1 to a. However, because a_n bounces between -1 and 1, by just letting $\epsilon = 1/2$ it is evident that (a_n) will not converge to a.

(b). Let (a_n) be the sequence $(0, 1, 0, 2, 0, 3, 0, 4, 0, 5, 0, 6, \dots)$ and $a = -2$. Then, let $\epsilon = 3$. So, for any $N \in \mathbb{N}$ choose $n = 2N + 1$ (or any other odd number larger than N, so that $a_n = 0$). Then we have $|a_n - a| = |0 - (-2)| = 2 < \epsilon$. So (a_n) does *Non*verge-type-2 to a. However, clearly $a_n \not\to -2$ since if, say, $\epsilon = 1$, and any N you choose, if n is larger than n and even then we certainly do not have $|a_n - a| < \epsilon$.

(c). Let $a_n = n$ and $a = 2$. Then for any $\epsilon > 0$, let $N = 1$ and $n = 2$. Then $|a_n - a| = |2 - 2| = 0 < \epsilon$. So (a_n) does *Non*verge-type-3 to a. However, because a_n is going off to ∞, clearly it is not converging to 2. Further, it does not *Non*verge-type-2 to 2, because for any $\epsilon > 0$, let $N = \lceil \epsilon + 10 \rceil$. Then, for every $n > N$, note that

$$|a_n - 2| = |\lceil \epsilon + 10 \rceil - 2| \geq \epsilon + 10 - 2 > \epsilon.$$

(d). This is a bad definition of convergence because, by letting $N = 1$, if forces $|a_n - a| < \epsilon$ for every $n > N$. And that forces $a_n = a$ for every $n > 2$. And that means that the only sequences that would "converge" under this definition are constant sequences and sequences which are constant starting with the second term. That is,

$$a, a, a, a, a, \dots$$

and

$$b, a, a, a, a, \dots$$

both converge to a, but nothing else does. □

Solution of Question 3.
(a). Let $\epsilon > 0$. Let $N = \frac{4}{\epsilon^2}$ and $n > N$. Then,

$$|a_n - a| = \left|8 + \frac{2}{\sqrt{n}} - 8\right| = \left|\frac{2}{\sqrt{n}}\right| = \frac{2}{\sqrt{n}} < \frac{2}{\sqrt{N}} = \frac{2}{\sqrt{4/\epsilon^2}} = \frac{2}{2/\epsilon} = \epsilon.$$

That is, $|a_n - a| < \epsilon$ for any $n > N$. And so, by definition, $a_n \to 8$ as $n \to \infty$. □

(b). Let $\epsilon > 0$. Let $N = \frac{39}{12\epsilon} + \frac{3}{4}$ and let $n > N$. Then,

$$\begin{aligned}
|a_n - a| &= \left|\frac{5n+6}{4n-3} - \frac{5}{4}\right| \\
&= \left|\frac{4(5n+6)}{4(4n-3)} - \frac{5(4n-3)}{4(4n-3)}\right| \\
&= \left|\frac{20n+24}{16n-12} - \frac{20n-15}{16n-12}\right| \\
&= \left|\frac{39}{16n-12}\right|
\end{aligned}$$

which, because $n > 0$,

$$= \frac{39}{16n-12},$$

and since $n > N$,

$$\begin{aligned}
&< \frac{39}{16N-12} \\
&= \frac{39}{16\left(\frac{39}{16\epsilon} + \frac{3}{4}\right) - 12} \\
&= \frac{39}{39/\epsilon} \\
&= \epsilon.
\end{aligned}$$

That is, $|a_n - a| < \epsilon$ for any $n > N$. And so, by definition, $a_n \to \frac{5}{4}$ as $n \to \infty$. □

(c). Let $\epsilon > 0$. Let $N = \frac{1}{\epsilon^2}$ and let $n > N$. Then,

$$\begin{aligned}
|a_n - 0| &= \left|\frac{\sqrt{n}}{n+\sqrt{n}} - 0\right| \\
&= \frac{\sqrt{n}}{n+\sqrt{n}} \\
&< \frac{\sqrt{n}}{n} \\
&= \frac{1}{\sqrt{n}}.
\end{aligned}$$

Since $n > N = 1/\epsilon^2$, we have

$$\begin{aligned}
\frac{1}{\sqrt{n}} &< \frac{1}{\sqrt{N}} \\
&= \frac{1}{\sqrt{1/\epsilon^2}} \\
&= \epsilon.
\end{aligned}$$

That is, $|a_n - 0| < \epsilon$ for any $n > N$. And so, by definition, $a_n \to 0$ as $n \to \infty$. □

(d). Let $\epsilon > 0$, let $N = \max\{\frac{5}{4\epsilon} + \frac{3}{2}, 1\}$, and notice that this implies $N \geq \frac{5}{4\epsilon} + \frac{3}{2}$ and $N \geq 1$. Then, consider any $n > N$. First, observe that the function can be simplified by factoring.

$$\begin{aligned}
|a_n - \tfrac{1}{2}| &= \left|\frac{n^2+2n+1}{2n^2-n-3} - \frac{1}{2}\right| \\
&= \left|\frac{(n+1)(n+1)}{(2n-3)(n+1)} - \frac{1}{2}\right| \\
&= \left|\frac{n+1}{2n-3} - \frac{1}{2}\right| \\
&= \left|\frac{2(n+1)-(2n-3)}{2(2n-3)}\right| \\
&= \left|\frac{5}{4n-6}\right|
\end{aligned}$$

and because $n > N \geq 1$,

$$= \frac{5}{4n-6}.$$

Now, because $N \geq \frac{5}{4\epsilon} + \frac{3}{2}$ and $n > N$, we have

$$|a_n - \tfrac{1}{2}| = \frac{5}{4n-6} < \frac{5}{4N-6} = \frac{5}{4(\frac{5}{4\epsilon}+\frac{3}{2})-6} = \frac{5}{5/\epsilon} = \epsilon.$$

That is, $|a_n - \frac{1}{2}| < \epsilon$ for any $n > N$. And so, by definition, $a_n \to \frac{1}{2}$ as $n \to \infty$. □

Solution of Question 4. Part (a). Let $\epsilon > 0$. Since $\frac{\epsilon}{2} > 0$ and $a_n \to a$, we know that there exists some N_1 such that $|a_n - a| < \epsilon/2$ for all $n > N_1$. Likewise, since $\frac{\epsilon}{2} > 0$ and $b_n \to b$, we know that there exists some N_2 such that $|b_n - b| < \epsilon/2$ for all $n > N_2$.

Now let $N = \max\{N_1, N_2\}$. Then for any $n > N$ we have

$$\begin{aligned}
|(a_n+b_n)-(a+b)| &= |(a_n-a)+(b_n-b)| \\
&\leq |a_n-a|+|b_n-b| && \text{(triangle inequality)} \\
&< \frac{\epsilon}{2}+\frac{\epsilon}{2} \\
&= \epsilon,
\end{aligned}$$

completing the proof. □

Part (b). Let $\epsilon > 0$. Since $\frac{\epsilon}{|c|+1} > 0$ and $a_n \to a$, we know that there exists some N such that $|a_n - a| < \frac{\epsilon}{|c|+1}$ for all $n > N$. Therefore, for this same N we have that

$$|c\cdot a_n - c\cdot a| = |c(a_n - a)| = |c|\cdot|a_n - a| < |c|\cdot\frac{\epsilon}{|c|+1} < \epsilon,$$

for all $n > N$. Therefore $(c \cdot a_n)$ converges to $c \cdot a$. □

Solution to Question 5. Part (a). Answers vary. But one example is if (a_n) and (b_n) are both the sequence (n), which diverges to ∞, but their quotient is the constant sequence (1), which converges to 1. □

Part (b). One way is to let the denominator converge to zero, which blows up the fraction. So if (a_n) is the constant sequence (1), and (b_n) is the sequence $(1/n)$, then then both converge, but $(\frac{a_n}{b_n}) = (n)$, which diverges to ∞. □

Part (c). One way to achieve this is use the power of the squaring function to turn negative things positive. For example, letting $a_n = (-1)^n$ gives a sequence whose limit does not exist. But (a_n^2) is the sequence $1, 1, 1, 1, \ldots$, which converges to 1. □

Solution to Question 6.
Scratch Work. To have it be bounded we want part of the sequence to go off to ∞ (or $-\infty$), but to avoid it diverging to infinity we can not let the whole sequence get bigger and bigger. Remember, to diverge to ∞ means that for every $M > 0$ there exists an N such that for *every* $n > N$, we have $a_n > M$. Whereas to be unbounded you just have to have *some* $a_n > M$.

So that's the difference. For each M we want to get a term above M, we don't want *all* the terms to stay above M. There are many ways to do this, but below is one example.

Solution. There are many examples, but here's one:

$$a_n = \begin{cases} n & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases}$$

□

Solution to Question 7. Part (a). Let $a_n = 1 + \frac{(-1)^n}{n^\pi}$. Then note that

$$-\frac{1}{n^\pi} \leq \frac{(-1)^n}{n^\pi} \leq \frac{1}{n^\pi}.$$

Adding 1 to all sides gives

$$1 - \frac{1}{n^\pi} \ \leq \ a_n \ \leq \ 1 + \frac{1}{n^\pi}.$$

Now, provided that we can show that $\lim_{n\to\infty} \left(1 - \frac{1}{n^\pi}\right) = 1$ and $\lim_{n\to\infty} \left(1 + \frac{1}{n^\pi}\right) = 1$, then by the Sequence Squeeze Theorem (Theorem 3.23) we will be able to conclude that

$$\lim_{n\to\infty} a_n = 1,$$

which will complete the proof.

To do this, let $\epsilon > 0$. Define $b_n = 1 - \frac{1}{n^\pi}$. Then

$$\begin{aligned} |b_n - 1| &= \left|1 - \frac{1}{n^\pi} - 1\right| \\ &= \left|-\frac{1}{n^\pi}\right| \\ &= \frac{1}{n^\pi}. \end{aligned}$$

Choose $N = \frac{1}{\epsilon^{1/\pi}}$. If $n > N$, then

$$|b_n - 1| = \frac{1}{n^\pi} < \frac{1}{N^\pi} = \frac{1}{(1/\epsilon^{1/\pi})^\pi} = \frac{1}{1/\epsilon} = \epsilon.$$

Therefore, by definition, $\lim_{n\to\infty} b_n = 1$.

And, in almost the exact same way we can show that $\lim_{n\to\infty} \left(1 + \frac{1}{n^\pi}\right) = 1$. □

□

Part (b). This overly-elaborate problem is supposed to be a more interesting version of Example 3.28 in the book. The proof of it is nearly identical.

Let a_n be the the number containing the first n digits (after the decimal point) of this infinite-book translation into the decimal number.

Note that a_{n+1} and a_n match exactly until the last digits of a_{n+1}. Therefore, $a_{n+1} - a_n$ is a number with a bunch of zeros followed by a single digit in the $n+1$ place. In particular,

$$a_{n+1} - a_n > 0$$

for all n. We have shown that $a_{n+1} > a_n$ for all n, proving that (a_n) is monotone increasing. Furthermore note $a_n \leq 1$ for all n, since these are all decimal number starting "0.N".

We have shown that (a_n) is monotone increasing and bounded above, therefore by the monotone convergence theorem (Theorem **??**) the sequence converges to a book telling the entire history of time. $\square$

Solution to Question 8. One such sequence is defined by

$$a_n = \begin{cases} -3 & \text{if } n = 3k \text{ for some } k \in \mathbb{Z} \\ 14 & \text{if } n = 3k+1 \text{ for some } k \in \mathbb{Z} \\ n & \text{if } n = 3k+2 \text{ for some } k \in \mathbb{Z} \end{cases}$$

This sequence has a subsequence $(-3, -3, -3, -3, -3, \dots)$ which converges to -3. It has another subsequence $(14, 14, 14, 14, 14, \dots)$ which converges to 14. And has another subsequence $(2, 5, 8, 11, 14, \dots)$ which diverges to ∞.

Solution to Question 9. Let (a_n) be a sequence of real numbers. Prove that if *every* subsequence of (a_n) converges, then (a_n) converges too.

Solution. Observe that by the definition of a subsequence (where $n_k = k$ for all k), (a_n) is a subsequence of (a_n). Therefore, since *every* subsequence converges, that would include (a_n) itself! So, indeed, we have shown that (a_n) converges. $\square$

Solution to Question 10. Every bounded sequence has a convergent subsequence.

Solution to Question 11. A sequence (a_n) is *Cauchy* if for all $\epsilon > 0$ there exists[1] some N such that

$$|a_m - a_n| < \epsilon$$

for all $m, n > N$.

Solution to Question 12. Let $\epsilon > 0$. By the Archimedean principle, there exists $N \in \mathbb{N}$ such that

$$N > \frac{2}{\epsilon}.$$

(Or, equivalently, there is an $N \in \mathbb{N}$ for which $\frac{1}{N} < \frac{\epsilon}{2}$.)

Let $m, n > N$. Then, using the triangle inequality,

$$\left|\frac{1}{n} - \frac{1}{m}\right| \leq \left|\frac{1}{n}\right| + \left|\frac{1}{m}\right| = \frac{1}{n} + \frac{1}{m} < \frac{1}{N} + \frac{1}{N} = \frac{2}{N} < \epsilon.$$

Therefore, for every $\epsilon > 0$ there exists N such that for all $m, n > N$ we have

$$\left|\frac{1}{n} - \frac{1}{m}\right| < \epsilon.$$

Hence $(1/n)$ is a Cauchy sequence. $\square$