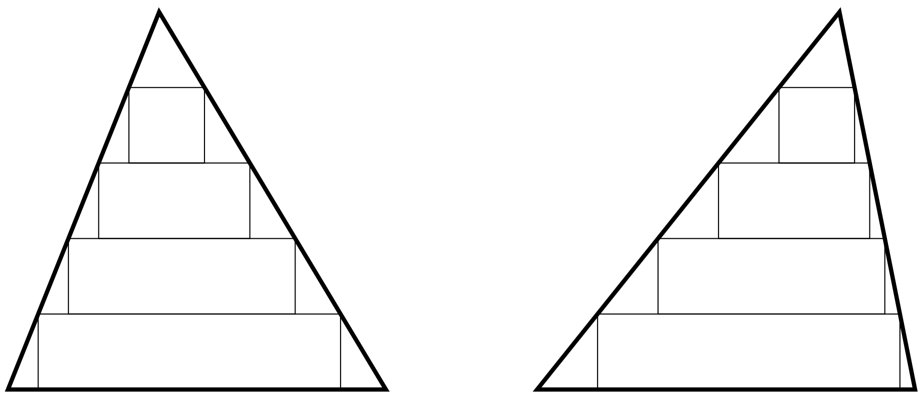
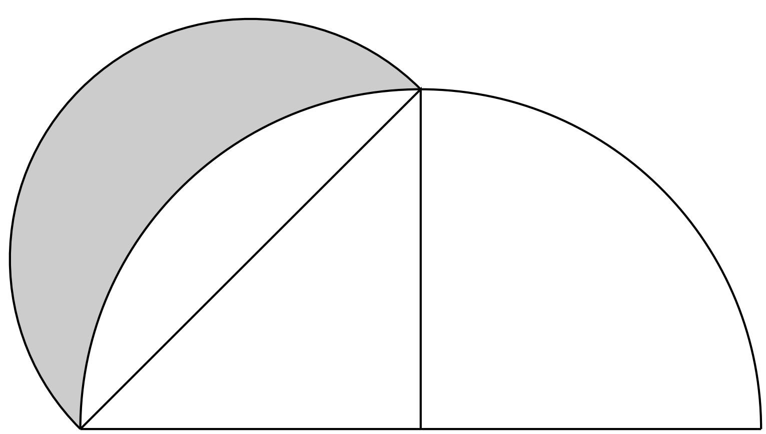
Chapter 7 Solved Exercises

Question 1(a). (Similar to Exercise 7.1) The method of exhaustion is not needed for polygons, but they do still make for good exercises and help to aid understanding. In this spirit, you will prove below that two triangles with the same base and height must have the same area.

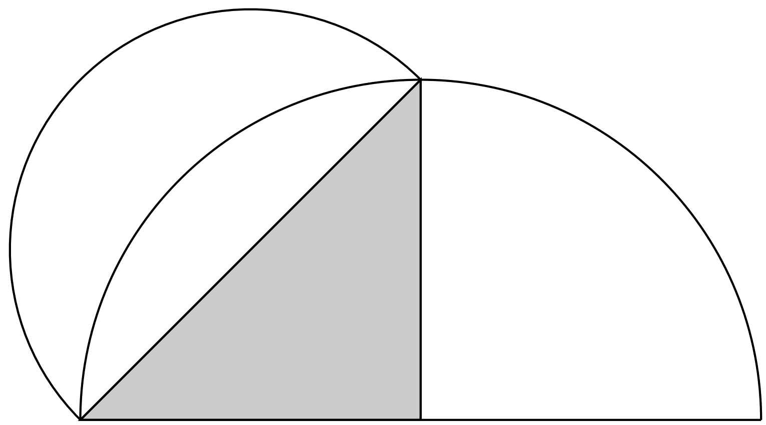
Suppose two triangles have the same base and height. Then, suppose you stack n rectangles — all of equal height and maximum width — inside each of the triangles. Show that the two sets of rectangles are identical. (As $n \to \infty$, the total area of these rectangles approaches each triangle's area, so by doing this you will have shown that the two triangles have the same area.)



Question 2. (Exercise 7.2) Before Archimedes used his method of exhaustion to conquer curvature, Hippocrates of Chios found the exact area of a *lune*, which is made from two semicircles and is pictured below.



He did this by showing it has the same area as the shaded triangle below.



In this exercise, you will prove this fact using the following steps.

(a) Redraw the above picture on your paper; you can include either or both of the shadings if you wish. Assign a radius r to the larger semicircle and write down this radius r on every line of that length in the picture.

(b) Using knowledge of triangles, determine the diameter and then the radius of the smaller semicircle.

(c) Thinking now about area, determine how many times larger the big semicircle is compared to the small semicircle. (Not just the shaded portion, the semicircles in their entirety.)

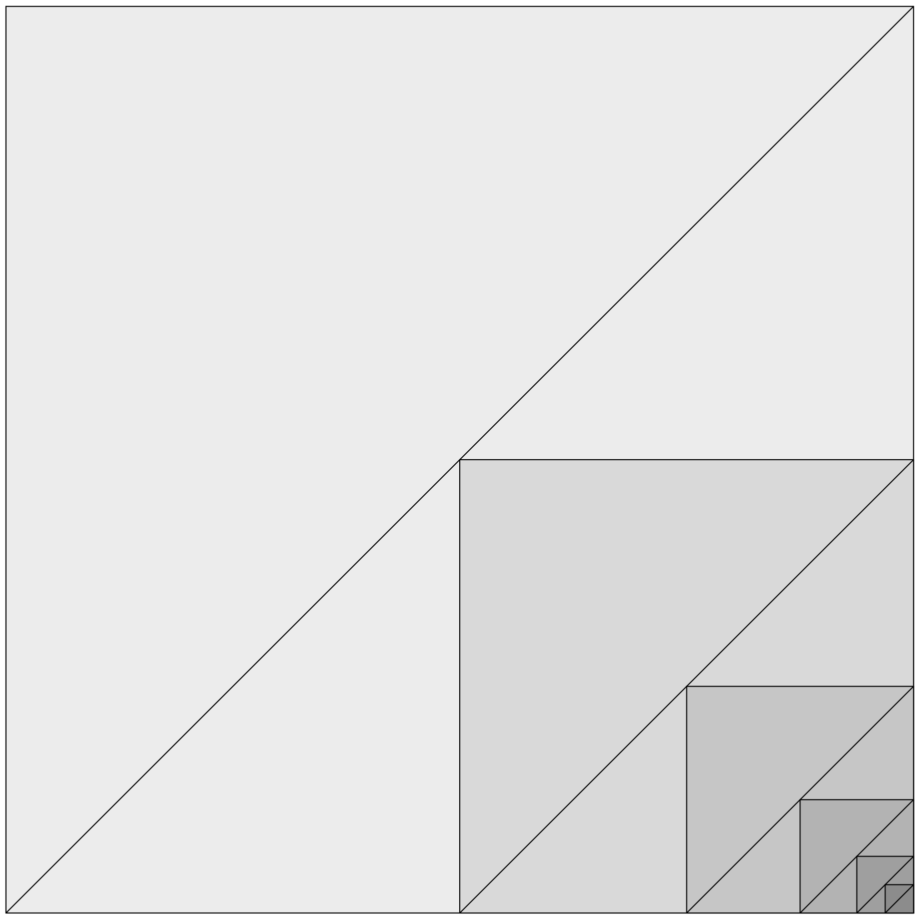
(d) Notice that only half of the larger semicircle is being shaded at all. How does the area of half of the large semicircle compare to the area of the entire small semicircle?

(e) The two semicircles are not entirely shaded, but note that the non-shaded portions overlap. Using this observation, conclude the proof.

Question 3. (Similar to Exercise 7.5) In this chapter we saw Archimedes' proof-by-picture that

$$1 = \frac{3}{4} + \frac{3}{16} + \frac{3}{64} + \frac{3}{256} + \dots.$$

Explain how Archimedes could have instead used the following picture to prove this result.

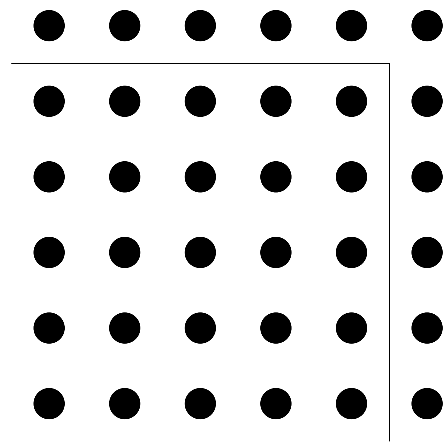


Question 4. (Similar to Exercise 7.7) This chapter discussed an early example of a proof by induction, which came from the Kerala school. Perhaps the earliest implicit proof by induction is due to Baghdadi mathematician Abū Bakr al-Karaji (953-1029). He used this technique for a few different proofs, one of which is of the neat fact that for every $N \in \mathbb{N}$,

$$1 + 3 + 5 + \cdots + (2N - 1) = N^2.$$

Similar to our example from Kerala, al-Karaji's proofs by induction did not have the formality of today's proofs by induction. In most, he shows how a specific case implies the next case. Nevertheless, his argument is clearly communicating the core ideas of induction.

(a) Below is one way to argue that the fifth case implies the sixth case. Explain why.



(b) Write out a formal proof by induction of this fact, including a base case, inductive hypothesis and induction step.

Question 5. (Exercise 7.9(a)) Using the Internet, a book or an original source, find an early proof by induction that was not included in this chapter or the exercises. Explain the proof thoroughly and discuss its history.

Question 6. (Exercise 7.9(c)) Using the Internet, a book or an original source, write about Pierre de Fermat's use of the method of infinite decent, and how it relates to induction.

Question 7. (Exercise 7.10) Assume that $0 < r < 1$. In this chapter we saw how Madhava of Sangamagrama proved that

$$1 + r + r^2 + r^3 + \cdots = \frac{1}{1-r}.$$

Assume that $a > 0$. Prove that

$$a + ar + ar^2 + ar^3 + \cdots = \frac{a}{1-r}$$

by adapting Madhava of Sangamagrama's proof. (That is, do not simply multiply both sides by a. Show how the original proof could be redone to arrive at this conclusion.)

Question 8. (Exercise 7.15) Archimedes lived in Syracuse, on the island of Sicily, around 240 BC. Research what life was like in Syracuse around this time and write a 300-word essay on what you learned.

Question 9. (Exercise 7.17) Madhava of Sangamagrama founded the Kerala school of astronomy and mathematics. Research this school and write a 500-word essay on what you learned.

Question 10. (Exercise 7.21) Prove Archimedes' favorite theorem using his mechanical method. That is, prove that if you circumscribe a cylinder around a sphere, then the sphere's volume is two-thirds that of the cylinder. You may look up the proof online or in a book, but the proof you write down should be presented thoroughly and at a level that your classmates can follow along.