Chapter 5 Solved Exercises

Question 1. (Similar to Exercise 5.1) We are halfway through the main content of this book, so it is a good time to reflect. Has your perspective on math or math history changed while reading this book? Write a reflection.

Question 2. (Exercise 5.4) The proof of Theorem 5.1 used the fact that if n is an integer and 2 divides n^2, then 2 must also divide n. In this exercise you will be guided through a proof-by-contrapositive of this fact. In it, you may use the fact that every integer can either be written in the form $2k$ (if it is even) or $2k+1$ (if it is odd), where k is an integer.

(a) Assume that 2 does not divide n, which means that $n = 2k+1$ for some integer k. That is, n is assumed to be odd. Show that n^2 will also be odd.

(b) Explain why the contrapositive concludes the proof.

Question 3. (Exercise 5.7) In Joseph-Louis Lagrange's 1898 book *Lectures on Elementary Mathematics*, he gave the following two-sentence proof sketch that $\sqrt{2}$ is irrational.

> It's impossible to find a whole number which multiplied by itself will give 2. It cannot be found in fractions, for if you take a fraction reduced to its lowest terms, the square of this fraction will again be a fraction reduced to its lowest terms, and consequently cannot be equal to the whole number 2.

Fill in the details of this argument.

Question 4. (Similar to Exercise 5.10) Here is another proof sketch that $\sqrt{2}$ is irrational. This one is due to Gustave Robson. Fill in the details.

> When working in base-10, a perfect square's final digit can only be 0, 1, 4, 5, 6 or 9. Meanwhile, twice a square must end with a 0, 2 or 8. Thus, in the equation $a^2 = 2b^2$, where a and b are integers, both a and b must be divisible by 5. Hence, $\frac{a}{5}$ and $\frac{b}{5}$ must also satisfy the equation. Hence, $\frac{a}{5^2}$ and $\frac{b}{5^2}$ must also satisfy the equation. And so on, leading to a contradiction.

Question 5. (Exercise 5.23) There are many ways to prove the infinitude of primes, which the next two exercises will explore. In this exercise, we will prove the theorem by using *Fermat numbers*, which we discussed on page **??** in Chapter **??** when discussing Gauss' construction of a 17-gon using a straightedge and compass. The *Fermat number* F_n is the number $2^{2^n}+1$. Here are the first four: $F_0 = 3$, $F_1 = 5$, $F_2 = 17$ and $F_3 = 257$.

(a) Prove by induction that for every $n \in \mathbb{N}$,

$$F_0 \cdot F_1 \cdot F_2 \cdot \ldots \cdot F_{n-1} = F_n - 2.$$

(b) Let $m, n \in \mathbb{N}_0$, and assume $n \neq m$. Using part (a), prove that F_m and F_n are relatively prime.

(c) Using part (b), and considering the infinite sequence $F_1, F_2, F_3, \ldots$, prove that there must be infinitely many primes.

Question 6. (Similar to Exercise 5.24) Below you will prove that there are infinitely many primes.

(a) Using a book or the internet, look up what *Bertrand's postulate* says. Write it down.

(b) Explain how this postulate proves that there are infinitely many primes.

Question 7. (Similar to Exercise 5.28) Consider the following problem.

> There are certain things whose number is unknown. If we count them by twos, we have one left over; by threes, we have one left over; and by sevens, four are left over. How many things are there?

Explain how Sun Zi would have solved this. You may use modern notation.

Question 8. (Exercise 5.37) Show how Fermat used his method of infinite descent to prove the $n = 4$ case of Fermat's last theorem. You may use any book or internet source, but the solution you present should be presented at a level that your classmates can follow along with each step.

Question 9. (Exercise 5.40) Write a 500-word essay on how unsolved problems affect the direction of mathematics and of math history.

Question 10. (Exercise 5.42) Come up with your own exercise that uses the information from this chapter. Make sure it is different from the other exercises in this chapter and have its level of difficulty be at about the same level as the others. Then, answer your question.