# Chapter 3 Solved Exercises

- You must prove all of your answers (unless stated otherwise).
- Remember the proof techniques that you learned in Math 108: Direct, by contradiction, by using the contrapositive, by induction, and/or by cases. And you can disprove something by exhibiting a counterexample.

**Question 1.** State the definition (using $\epsilon$ and $N$) for a sequence $(a_n)$ to converge to a point $a$.

**Question 2.** (Similar to Exercise 3.3) This problem is to help you get a feel for the quantifiers in the definition of convergence that you wrote down in problem 1. Your task: For each of the following definitions of *Non*verges, give an example a sequence $(a_n)$ and a value $a$ for which

- $(a_n)$ does not converge to $a$ (based on the real definition from Question 1 above),
- $(a_n)$ does *Non*verge to $a$ based on the definition given.

Give a different example for each problem. For each of them, explain why your example works in a few sentences (no need to prove it completely). Have a different example in each case. Your example for *Non*verges-type-3 should <u>not</u> work for *Non*verges-type-2.

1. Definition 1: The sequence $(a_n)$ *Non*verges-type-1 to $a$ if for all $\epsilon > 0$ and for all $N \in \mathbb{N}$, there exists some $n > N$ such that $|a_n - a| < \epsilon$.
2. Definition 2: The sequence $(a_n)$ *Non*verges-type-2 to $a$ if there exists some $\epsilon > 0$ such that for all $N \in \mathbb{N}$ there exists some $n > N$ such that $|a_n - a| < \epsilon$.
3. Definition 3: The sequence $(a_n)$ *Non*verges-type-3 to $a$ if there exists some $\epsilon > 0$ and there exists some $N \in \mathbb{N}$ such that for some $n > N$ we have $|a_n - a| < \epsilon$.

Finally, explain why this is not a good definition of convergence:

4. Definition 4: The sequence $(a_n)$ *Non*verges-type-4 to $a$ if for all $\epsilon > 0$ and for all $N \in \mathbb{N}$ and for all $n > N$, we have $|a_n - a| < \epsilon$.

**Question 3.** (Similar to Exercise 3.4)

(a) Let $a_n = 8 + \dfrac{2}{\sqrt{n}}$. Show that $a_n \to 7$ as $n \to \infty$.

(b) Let $a_n = \dfrac{5n+6}{4n-3}$. Show that $\lim\limits_{n\to\infty} a_n = \dfrac{5}{4}$.

(c) Let $a_n = \dfrac{\sqrt{n}}{n+\sqrt{n}}$. Show that $a_n \to 0$.

(d) Let $a_n = \dfrac{n^2+2n+1}{2n^2-n-3}$. Show that $(a_n)$ converges to $\frac{1}{2}$.

**Question 4.** (Exercise 3.8) Assume that $(a_n)$ converges to some $a \in \mathbb{R}$ and $(b_n)$ converges to some $b \in \mathbb{R}$. Also assume $c \in \mathbb{R}$.

(a) Prove that $(a_n + b_n)$ converges to $a + b$.

(b) Prove that $(c \cdot a_n)$ converges to $c \cdot a$.

**Question 5.** (Similar to Exercise 3.16)

(a) Give an example of two different divergent sequences $(a_n)$ and $(b_n)$ for which $(\frac{a_n}{b_n})$ converges.

(b) Give an example of two convergent sequences $(a_n)$ and $(b_n)$ for which $(\frac{a_n}{b_n})$ diverges.

(c) Give an example of a divergent sequence $(a_n)$ for which $(a_n^2)$ converges. (That is, we are squaring each term of $a_n$.)

**Question 6.** (Exercise 3.27) Give an example of an unbounded divergent sequence whose terms are all positive and whose limit does not exist. (Recall, there are three types of divergence: (1) diverging to $\infty$, (2) diverging to $-\infty$, or (3) does not exist. Your sequence should be of this third type.). You do not need to prove your answer.

**Question 7.** (Similar to Exercise ???) Prove that the following sequences converge.

(a) Let $a_n = 1 + \frac{(-1)^n}{n^\pi}$.

(b) Imagine one encoded the lowercase letters as $a = 01$, $b = 02$, $c = 03$, ..., $z = 26$; the uppercase letters $A = 27$, $B = 28$, $C = 29$, ..., $Z = 52$; the ten digits as $0 = 53$, $1 = 54$, $2 = 55$, ..., $9 = 62$; the 32 punctuation/symbols on a standard keyboard as the numbers 63, 64, ..., 95; and, finally, a space as 96.

With this, the entire keyboard is translated into two-digit numbers, and one is able to translate any text into a sequence of two-digit numbers. For example, "Math is cool" can be written as

39 01 20 08 96 09 19 96 03 15 15 12

In fact, one can even write it as a single number between 0 and 1 by appending numbers after the decimal point. For example, "Math is cool" translates to

0.390120089609199603151512

Next, imagine that universe's lifespan is infinite, and that God exists, and that He wrote an infinite book telling the entire infinite story of universe, with every detail included, such as the life story of every human, including your own. Prove that there exists some number between 0 and 1 that tells the entire infinite story of the universe.

**Question 8.** (Similar to Exercise 3.31) Give an example of a sequence $(a_n)$ which has:

- A subsequence converging to -3,
- Another subsequence converging to 14,
- And another subsequence diverging to $\infty$.

Give a brief explanation for why your example works.

**Question 9.** (Exercise 3.32) Let $(a_n)$ be a sequence of real numbers. Prove that if *every* subsequence of $(a_n)$ converges, then $(a_n)$ converges too.

**Question 10.** State the Bolzano-Weierstrass theorem.

**Question 11.** Define what it means for a sequence to be *Cauchy*.

**Question 12.** Using the triangle inequality and the Archimedean principle, prove that the sequence $(1/n)$ is Cauchy.