Chapter 2 Solutions to Selected Exercises

Notes:

- The questions are in a separate PDF on LongFormMath.com.
- For most problems there are many correct solutions, so the below are not the only correct ways to solve the problems.
- If you spot an error, please email it to me at LongFormMath@gmail.com. Thanks!

Solution to Question 1.

(a) The power set of $\{a, ☺, 7\}$ is

$$\mathcal{P}(\{a, ☺, 7\}) = \{\ \{a, ☺, 7\},\ \{a, ☺\},\ \{a, 7\},\ \{☺, 7\},\ \{a\},\ \{☺\},\ \{7\},\ \varnothing\ \}.$$

(b) The power set of $\{y\}$ is

$$\mathcal{P}(\{y\}) = \{\ \{y\},\ \varnothing\ \}.$$

(c) If a set has n elements, then its power set has 2^n elements. In other words,

$$|\mathcal{P}(S)| = 2^n \quad \text{whenever } |S| = n.$$

Solution to Question 2.

(a) A bijection $f_1 : (5, \infty) \to (11, \infty)$ is

$$f_1(x) = x + 6.$$

(b) A bijection $f_2 : (5, \infty) \to (-\infty, 11)$ is

$$f_2(x) = 16 - x.$$

(c) A bijection $f_3 : (0, \infty) \to (0, 1)$ is

$$f_3(x) = \frac{x}{x+1}.$$

(d) A bijection $f_4 : \mathbb{R} \to (0, \infty)$ is

$$f_4(x) = 2^x.$$

Solution of Question 3. Assume that $A \subseteq \mathbb{N}$. If A is finite, then we are done, so assume that A is infinite. Then the elements of A can be ordered in an infinite list as

$$a_1 < a_2 < a_3 < a_4 < \dots.$$

This is due to the well-ordering principle. By applying it to A, one obtains a_1. Then by applying it to $A \setminus \{a_1\}$, one gets a_2. And so on.

Now, consider the function

$$f : \mathbb{N} \longrightarrow A$$
$$f(n) = a_n.$$

Note that f is injective since, if $f(m) = f(n)$, then $a_m = a_n$, implying $m = n$. Also note that f is surjective, since for any $x \in A$, $x = a_m$ for some $m \in \mathbb{N}$, and hence $f(m) = a_m$. So f is a bijection from $\mathbb{N}$ to A which means that $|A| = |\mathbb{N}|$.

So indeed, either A is finite or $|A| = |\mathbb{N}|$. □

Solution of Question 4. Recall that an infinite set S is *countable* if $|S| = |\mathbb{N}|$. And note two more things: First, by Question 3 on this homework, $|\mathbb{N}|$ is the smallest infinity; and second, that by Fact 2.4 in the book, $|S| \geq |T|$ if and only if there is a surjection from S to T. Combining these, in order to prove that $A \cup B$ is countable, it suffices to show that there exists a surjection from $\mathbb{N}$ to $A \cup B$.

Since both A and B are infinite countable, by Definition 2.10 there are bijections $f : \mathbb{N} \to A$ and $g : \mathbb{N} \to B$. Define $h : \mathbb{N} \to A \cup B$ by

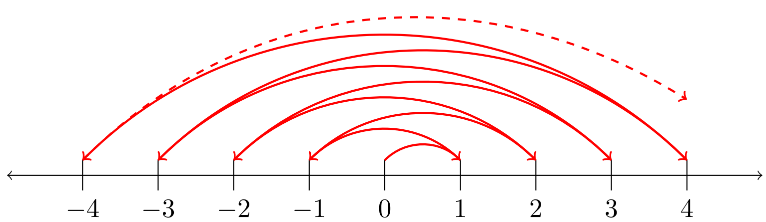
$$h(n) = \begin{cases} f\left(\frac{n}{2}\right), & \text{if } n \text{ is even,} \\ g\left(\frac{n-1}{2}\right), & \text{if } n \text{ is odd.} \end{cases}$$

The claim is that this function h is a surjection, and hence proves that $A \cup B$ is countable.

To see this, pick any $y \in A \cup B$. If $y \in A$, then because $f : \mathbb{N} \to A$ is a bijection, there exist some $n_0 \in \mathbb{N}$ for which $y = f(n_0)$. If, on the other hand, $y \notin A$, then it must be that $y \in B$. And in this case, we are assured that $y = g(m_0)$ for some $m_0 \in \mathbb{N}$, due to g being a bijection.

In the first case, where $y = f(n_0)$, note that $h(2n_0) = f(\frac{2n_0}{2}) = f(n_0) = y$. And in the second case, where $y = g(m_0)$, note that $h(2m_0 + 1) = g(\frac{2m_0+1-1}{2}) = g(m_0) = y$. That is, in either case, we have found an $n \in \mathbb{N}$ for which $h(n) = y$, proving that f is a surjection, and hence that $A \cup B$ is countable. □

Solution to Question 5. Intuitively, our bijection will do this:



One that does this is:

$$f : \mathbb{N} \longrightarrow \mathbb{Z}$$
$$f(n) = \begin{cases} n/2 & \text{if } n \text{ is even;} \\ -(n-1)/2 & \text{if } n \text{ is odd.} \end{cases}$$

□

Solution to Question 6. Let $\mathbb{I}$ be the set of irrational numbers. And recall that in Question 4 above we proved that the union of two countable sets is countable.

Now, assume for a contradiction that $\mathbb{I}$ is countable. Then, since $\mathbb{R} = \mathbb{Q} \cup \mathbb{I}$, and since we know that $\mathbb{Q}$ is countable, this would imply that $\mathbb{R}$ is a union of two countable sets, and hence is countable by Question 4. But Theorem 2.9 tells us that $\mathbb{R}$ is uncountable, giving the contradiction and completing the proof. □

Solution to Question 7. Let A be the collection of rational numbers with denominator $|n| \leq 10$. If A were dense in $\mathbb{R}$, then by definition there would be some m/n from A such that

$$0 < \frac{m}{n} < \frac{1}{10}.$$

However, among the positive elements in A (that is, when $m, n > 0$), if

$$\frac{m}{n} < \frac{1}{10}$$

then

$$10 \leq 10m < n,$$

implying that $10 < n$, which is a contradiction. $\square$

Solution to Question 8. Two sets have the same size if and only if there is a bijection between them.

Solution to Question 9. If S is an infinite set, then S is *countable* if $|S| = |\mathbb{N}|$. Otherwise S is *uncountable*.

Solution to Question 10. There is no set whose cardinality is strictly between that of the naturals and the reals.