Chapter 1 Solutions to Selected Exercises

Notes:

- The questions are in a separate PDF on LongFormMath.com.
- For most problems there are many correct solutions, so the below are not the only correct ways to solve the problems.
- If you spot an error, please email it to me at LongFormMath@gmail.com. Thanks!

Solution to Question 1. The error is moving from the third line to the fourth. We had assumed that $x = y$, which means that $x - y = 0$. It's true that

$$(x + y) \cdot 0 = y \cdot 0,$$

no matter what $x + y$ and y are. But you are never allowed to divide by zero, which is what's done to move to the the next step. So indeed, $x + y$ and y could be anything at all in line 3, and certainly do not have to be equal to each other as asserted in line 4.

Note: There could also be a problem in the final step. If $2y = y$, then you can not necessarily cancel a y from each side; again, what if $y = 0$? In fact, the only way that $2y = y$ is possible is if y does equal 0! □

Solution to Question 2 (a). By definition of the inequality, $a < b$ and $c < d$ mean that $b - a \in P$ and $d - c \in P$, where P is the set of positive elements of $\mathbb{F}$. We wish to show that $a + c < b + d$; that is, we wish to show that $(b + d) - (a + c) \in P$.

Recall that P is closed under addition. Therefore, since $b - a \in P$ and $d - c \in P$, also we have

$$(b - a) + (d - c) \in P$$

Rewriting,

$$(b + d) - (a + c) \in P,$$

as desired. □

Second Solution of (a). Since $a < b$, we have $a + c < b + c$. And since $c < d$, we have $b + c < b + d$. Putting this all together:

$$a + c < b + c < b + d.$$

□

Solution of (b). There are many solutions to this. A couple examples:

- Let $a = c = -10$ and $b = d = 2$. Then $a < b$ and $c < d$, however
$$ac = (-10)(-10) = 100 \not< 4 = (2)(2) = bd.$$
- Let $b = c = 0$, $a = -1$, $d = 2$. Then $a < b$ and $c < d$, however
$$ac = (-1)(0) = 0 \not< 0 = (0)(2) = bd.$$

□

Solution of Question 3 (a). Assume for a contradiction that $\sqrt{3}$ is rational. Then, there must be some nonzero integers p and q where

$$\sqrt{3} = \frac{p}{q}.$$

Moreover, we may assume that this fraction is written in *lowest terms*, meaning that p and q have no common divisors. Then,

$$\sqrt{3}q = p.$$

And by squaring both sides,

$$3q^2 = p^2.$$

Since $q^2 \in \mathbb{Z}$, by the definition of divisibility this implies that $3 \mid p^2$, and hence $3 \mid p$ (this was likely proved in your intro-to-proofs course; in my Proofs book it is Lemma 2.17 part (iii)). By a second application of the definition of divisibility, this means that $p = 3k$ for some nonzero integer k. Plugging this in,

$$\begin{aligned} 3q^2 &= p^2 \\ 3q^2 &= (3k)^2 \\ 3q^2 &= 9k^2 \\ q^2 &= 3k^2. \end{aligned}$$

Therefore, $3 \mid q^2$, and hence $3 \mid q$. But this is a contradiction: We had assumed that p and q had no common factors, and yet we proved that 3 divides each. Therefore $\sqrt{3}$ cannot be rational, meaning it is irrational. □

Solution of Question 3 (b). If you try to adapt it, you reach the point where you assert that $4 \mid p^2$ implies $4 \mid p$. But this is false. For example, if $p = 2$ or $p = 6$, this fails.

Solution of Question 4. We will prove this by using the contrapositive. Remember, a statement and its contrapositive are logically equivalent, so if you prove the contrapositive then you have also proven the original statement. Here is the statement that we were given to prove:

$$a < b + \epsilon \text{ for every } \epsilon > 0 \quad \implies \quad a \le b$$

And here is its contrapositive:

$$\text{not}(a \le b) \quad \implies \quad \text{not}(a < b + \epsilon \text{ for every } \epsilon > 0)$$

I.e.,

$$a > b \quad \implies \quad a \ge b + \epsilon \text{ for some } \epsilon > 0$$

So let's prove this instead by finding a specific ϵ that works.

Let $a > b$, which implies $a - b > 0$. Set $\epsilon = a - b$; note that $a = b + \epsilon$, so of course also $a \ge b + \epsilon$. Moreover, we have noted that $\epsilon > 0$. So with this ϵ,

$$a \ge b + \epsilon,$$

completing the proof. □

Second Solution. A second very similar solution uses proof by contradiction. Assume for a contradiction that $a > b$. Since $a < b + \epsilon$ for <u>all</u> $\epsilon > 0$, and since $a > b$ implies $a - b > 0$, we can set $\epsilon = a - b$ and the statement $a < b + \epsilon$ must still hold. However, for this choice of ϵ,

$$a < b + \epsilon = b + (a - b) = a.$$

So we have shown that $a < a$, which is a contradiction. □

Solution to Question 5. When you are trying to prove something holds for every positive integer, that is a good indication that induction is worth trying.

Base Case. When $n = 1$ the equality holds:

$$\sum_{k=1}^{1} \frac{1}{k(k+1)} = \frac{1}{1(1+1)} = \frac{1}{2} = \frac{1}{1+1}.$$

Inductive step. Assume that $\displaystyle\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$. We aim to use this inductive hypothesis to prove that $\displaystyle\sum_{k=1}^{n+1} \frac{1}{k(k+1)} = \frac{n+1}{(n+1)+1}$. By our assumption and algebra,

$$\begin{aligned}
\sum_{k=1}^{n+1} \frac{1}{k(k+1)} &= \left(\sum_{k=1}^{n} \frac{1}{k(k+1)}\right) + \frac{1}{(n+1)((n+1)+1)} \\
&= \left(\frac{n}{n+1}\right) + \frac{1}{(n+1)(n+2)} \qquad \text{(by the inductive hypothesis)} \\
&= \left(\frac{n(n+2)}{(n+1)(n+2)}\right) + \frac{1}{(n+1)(n+2)} \\
&= \frac{n^2+2n+1}{(n+1)(n+2)} \\
&= \frac{(n+1)(n+1)}{(n+1)(n+2)} \\
&= \frac{n+1}{(n+1)+1},
\end{aligned}$$

which completes the induction. □

Solution to Question 6. By the definition of a subset, it suffices to prove this: If $y \in f(A_1 \cap A_2)$, then $y \in f(A_1) \cap f(A_2)$. We will proceed in this way.

Pick any $y \in f(A_1 \cap A_2)$. Then, by the definition of the function f, there exists some $x \in A_1 \cap A_2$ such that $f(x) = y$. Since $x \in A_1$ we know that $f(x) = y \in f(A_1)$, and since $x \in A_2$ we know that $f(x) = y \in f(A_2)$. Since y is in both, it is in the intersection of the two:

$$y \in f(A_1) \cap f(A_2),$$

as desired. □

Solution to Question 7. If $\mathbb{F}$ is an ordered field (like $\mathbb{R}$) and if $x, y \in \mathbb{F}$, then

$$|x+y| \leq |x| + |y|.$$

□

Solution to Question 8. Let A be any nonempty subset of $\mathbb{N}$ which is bounded above. We aim to show that A has a supremum (i.e. a least upper bound), which will prove that $\mathbb{N}$ is complete.

If $m \in \mathbb{N}$ is an upper bound on A, then observe that the only possible elements of A are $\{1, 2, 3, \dots, m\}$. Therefore we may write $A = \{a_1, a_2, \dots, a_n\}$ where $n \leq m$. Moreover, assume that a_n is the largest element. That is, $a_i \leq a_n$ for all $i \in \{1, 2, \dots, n\}$.

We have noted that a_n is an upper bound on A, which is the first condition in the Supremums Analytically Theorem. Furthermore, for any $\epsilon > 0$, $a_n - \epsilon < a_n$. Notice that this satisfies the second condition in the

Supremums Analytically Theorem: it's true that for any $\epsilon > 0$ that there exists some $x \in A$ for which $a_n - \epsilon < x$, since $x = a_n$ always works!

So by the Supremums Analytically Theorem, $\sup(A) = a_n$. And since the supremum exists, we conclude that $\mathbb{N}$ is complete. □

Solution to Question 9. Let $A \subseteq \mathbb{R}$. Then $\sup(A) = \alpha$ if and only if

(i) α is an upper bound of A, and

(ii) Given any $\epsilon > 0$, $\alpha - \epsilon$ is *not* an upper bound of A. That is, there is some $x \in A$ for which $x > \alpha - \epsilon$.

Likewise, $\inf(A) = \beta$ if and only if

(i) β is a lower bound of A, and

(ii) Given any $\epsilon > 0$, $\beta + \epsilon$ is *not* a lower bound of A. That is, there is some $x \in A$ for which $x < \beta + \epsilon$.

Solution to Question 10. We will do one at a time. We will start with the more straightforward one.

Infimum Case. Let $A = \left\{\frac{n}{n+1} : n \in \mathbb{N}\right\}$. First we will show that $\frac{1}{2}$ is a lower bound of A. Since[1]

$$n \geq 1,$$

adding n to both sides gives

$$2n \geq n + 1,$$

and hence, by dividing over,

$$\frac{n}{n+1} \geq \frac{1}{2}.$$

So $1/2$ is a lower bound on A. The rest is just like Question 8's solution. For any $\epsilon > 0$, $\frac{1}{2} + \epsilon > \frac{1}{2}$. Notice that this satisfies the second condition in the Supremums Analytically Theorem: it's true that for any $\epsilon > 0$ there exists some $x \in A$ for which $\frac{1}{2} + \epsilon > x$, since $x = \frac{1}{2}$ always works! This concludes the infimum case.

Supremum Case. Let $A = \left\{\frac{n}{n+1} : n \in \mathbb{N}\right\}$. First we will show that 1 is an upper bound of A. Since

$$n + 1 \geq n$$

for all $n \in \mathbb{N}$, by dividing over,

$$1 > \frac{n}{n+1},$$

for all $n \in \mathbb{N}$. This shows that 1 is an upper bound on A. Next, we aim to show that $1 - \epsilon$ is not an upper bound on A for any $\epsilon > 0$. To this end, let $\epsilon > 0$. We need to show that there exists some n_0 such that

$$\frac{n_0}{n_0 + 1} > 1 - \epsilon.$$

The trick is to note that $\dfrac{n}{n+1} = \dfrac{(n+1) - 1}{n+1} = 1 - \dfrac{1}{n+1}$. Therefore the above is equivalent to

$$1 - \frac{1}{n_0 + 1} > 1 - \epsilon.$$

[1] I found this by doing scratch work first. I started with the final inequality and unwound it to the top line, which I knew was true.

To show this, note that by the Archimedean Principle, there is some $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < \epsilon$. This same n_0 then has the property that $\frac{1}{n_0+1} < \epsilon$. Thus, for this n_0, we have what we need:

$$\begin{aligned} \frac{1}{n_0} &< \epsilon \\ -\frac{1}{n_0} &> -\epsilon \\ 1 - \frac{1}{n_0+1} &> 1 - \epsilon. \end{aligned}$$

This completes the proof. □

Solution to Question 11 (a). Let $\epsilon = \sup(B) - \sup(A) > 0$. By the Supremums Analytically Theorem, there exists some $b \in B$ such that $b > \sup(B) - \epsilon$. By noting that $\sup(B) - \epsilon = \sup(A)$, this then implies that

$$b > \sup(A) \geq a,$$

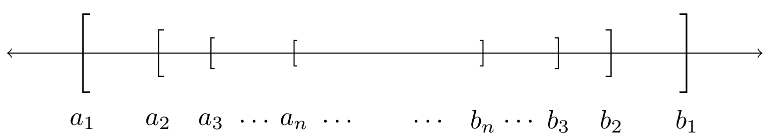
for every $a \in A$. That is, b is an upper bound of A.

Solution to Question 11 (b). There are many examples. Here are a couple:

- Let A and B both be the negative real numbers. Then $\sup(A) = \sup(B) = 0$. (This works just as well if A and B are both another set where the set's supremum is not contained in the set.)
- Let $A = \{1, 2, 3\}$ and $B = \{-4, 3\}$. (Or any other pair of finite sets which have the same largest element.)

□

Solution to Question 12. We are given a collection of nested intervals.



We aim to find some $x \in \mathbb{R}$ where $x \in I_n$ for all $n \in \mathbb{N}$. That is, some x for which $a_n \leq x \leq b_n$ for all $n \in \mathbb{N}$. To ensure that $a_n \leq x$ for all n, the smallest value that does this is the least upper bound of the set of the a_n's, which we can show exists:

Let

$$A = \{a_n : n \in \mathbb{N}\}$$

be the set of all the left end points of the intervals. First we note that each b_n serves as an upper bound on A. In particular, for any $m, n \in \mathbb{N}$, we are asserting that $a_m < b_n$. To see this, first note that if $n \leq m$ then

$$a_m \leq a_n < b_n.$$

And if $n > m$, then

$$a_m < b_m \leq b_n.$$

In either case we see that b_n is larger than a_m. And since m and n were arbitrary, each b_n is an upper bound of the set A. In summary, A is a subset of real numbers that is bounded above; thus by completeness, $\sup(A)$ exists. Let

$$x = \sup(A).$$

By the definition of the supremum, $a_n \leq x$ for all $n \in \mathbb{N}$. Moreover, since each b_n is an upper bound on A while x is the *least* upper bound on A, we see that $x \leq b_n$ for all $n \in \mathbb{N}$.

In summary, we know that $a_n \leq x \leq b_n$ for all $n \in \mathbb{N}$, completing the proof. □