Chapter 1 Solved Exercises

- You must prove all of your answers (unless stated otherwise).
- Remember the proof techniques that you learned in your intro-to-proofs course: Direct, by contradiction, by using the contrapositive, by induction, and/or by cases. And you can disprove something by exhibiting a counterexample.

Question 1. (Exercise 1.1) Explain the error in the following "proof" that $2 = 1$.
Let $x = y$. Then

$$\begin{aligned} x^2 &= xy \\ x^2 - y^2 &= xy - y^2 \\ (x+y)(x-y) &= y(x-y) \\ x + y &= y \\ 2y &= y \\ 2 &= 1. \end{aligned}$$

Question 2. (Exercise 1.12) Assume that $\mathbb{F}$ is an ordered field and $a, b, c, d \in \mathbb{F}$ with $a < b$ and $c < d$.

(a) Show that $a + c < b + d$.

(b) Prove that it is not necessarily true that $ac < bd$. Hint: prove this by finding a counterexample in the case that $\mathbb{F} = \mathbb{R}$.

Note whenever you use an axiom.

Question 3. (Exercise 1.9)

(a) Prove that $\sqrt{3}$ is irrational.

(b) What goes wrong when you try to adapt your argument from part (a) to show that $\sqrt{4}$ is irrational (which is absurd)?

Question 4. (Exercise 1.13) Let a, b and ϵ be elements of an ordered field. Show that if $a < b + \epsilon$ for every $\epsilon > 0$, then $a \leq b$.

Question 5. (Exercise 1.19) Prove that

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$$

for every positive integer n.

Question 6. (Exercise 1.21) Let $f : X \to Y$, and assume $A_1, A_2 \subseteq X$. Show that

$$f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2).$$

Question 7. State the triangle inequality. (Problems like these are meant to emphasize the importance of learning the statements of important definitions and theorems. When asked to state a theorem, make sure to state it fully, with the assumptions and conclusion.)

Question 8. (Exercise 1.26) Prove that $\mathbb{N}$ is complete.

Question 9. State the Suprema Analytically Theorem.

Question 10. (Exercise 1.29) Prove that

$$\sup\left(\left\{\frac{n}{n+1} : n \in \mathbb{N}\right\}\right) = 1 \quad \text{and} \quad \inf\left(\left\{\frac{n}{n+1} : n \in \mathbb{N}\right\}\right) = \frac{1}{2}.$$

Question 11. (Exercise 1.30) Assume $A, B \subseteq \mathbb{R}$ and that $\sup(A) < \sup(B)$.

(a) Show that there exists an element $b \in B$ that is an upper bound for A.

(b) Give an example to show that this is not necessarily the case if we instead only assume that $\sup(A) \leq \sup(B)$. You do not need to prove your answer.

Question 12. (Exercise 1.34) For each $n \in \mathbb{N}$, assume we are given a closed interval $I_n = [a_n, b_n]$. Also, assume that each I_{n+1} is contained inside of I_n. This gives a sequence of increasingly smaller intervals,

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq I_4 \supseteq \dots.$$

Prove that

$$\bigcap_{n=1}^{\infty} I_n \neq \emptyset.$$

That is, prove that there is some real number x such that $x \in I_n$ for every $n \in \mathbb{N}$.