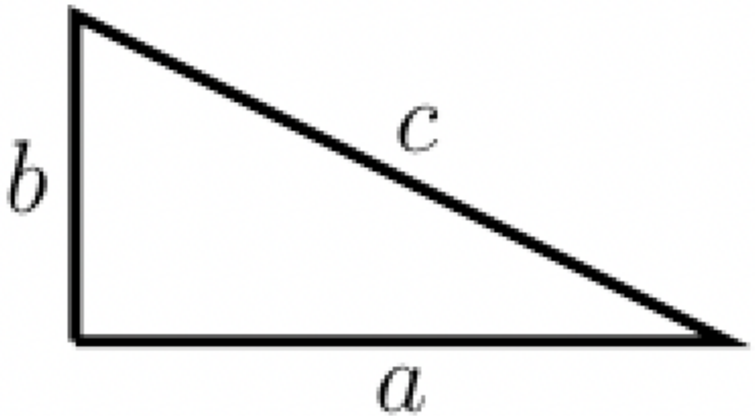
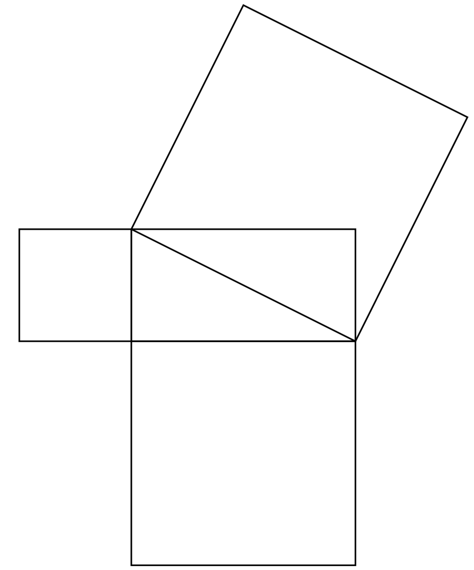
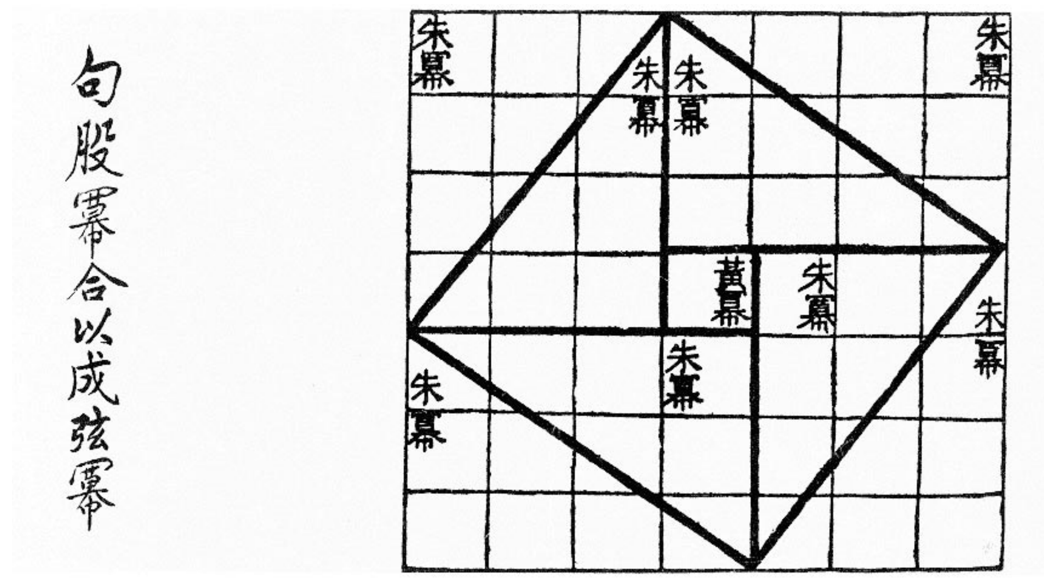
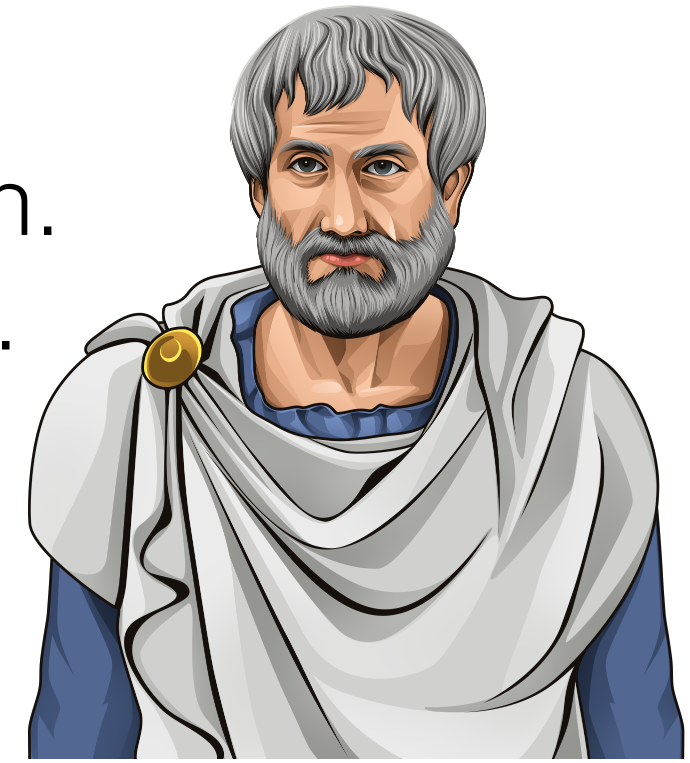






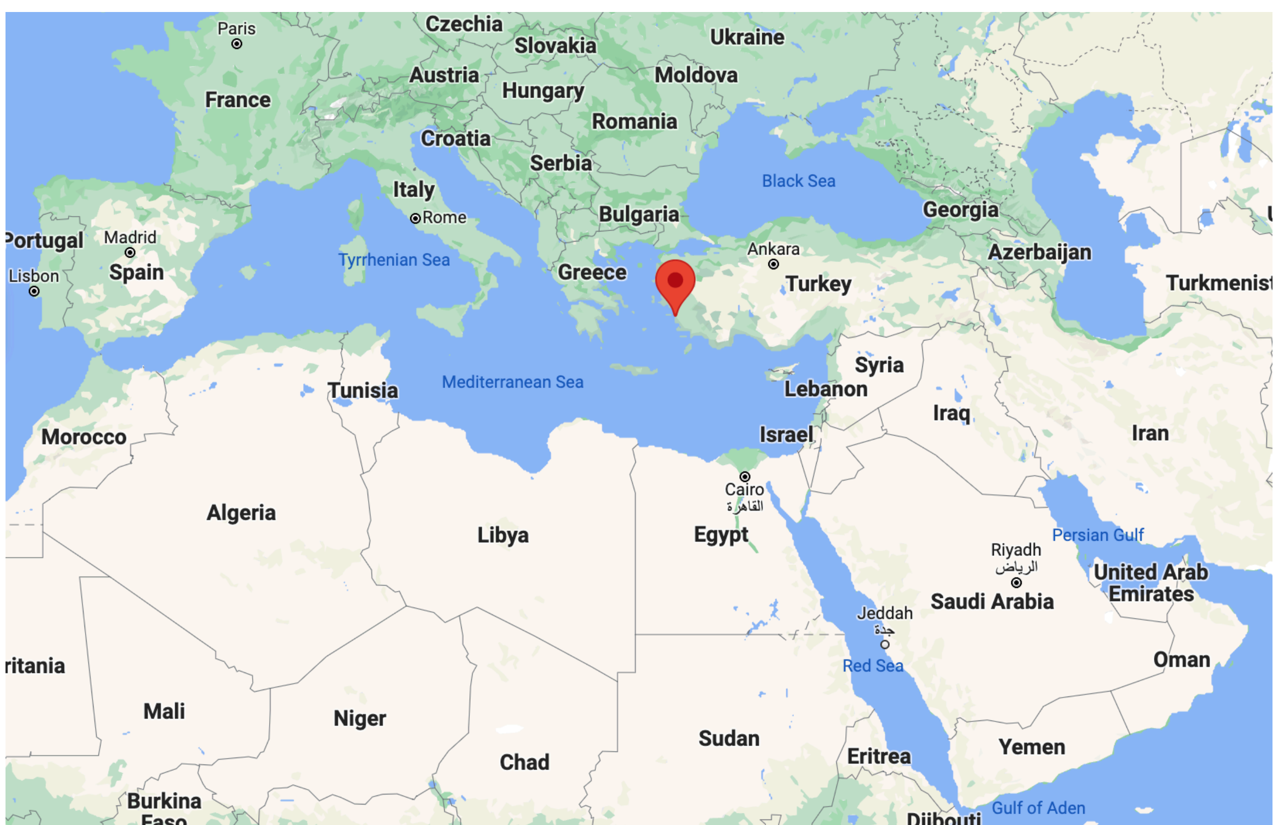
Chapter 3: Demonstrative Mathematics

Demonstrative Math



- Demonstrative math = Proof-based math. Before: *What* is true. After: *Why* it is true.

- Thales of Miletus (~624-545 BC, from Miletus) was a Greek mathematician.



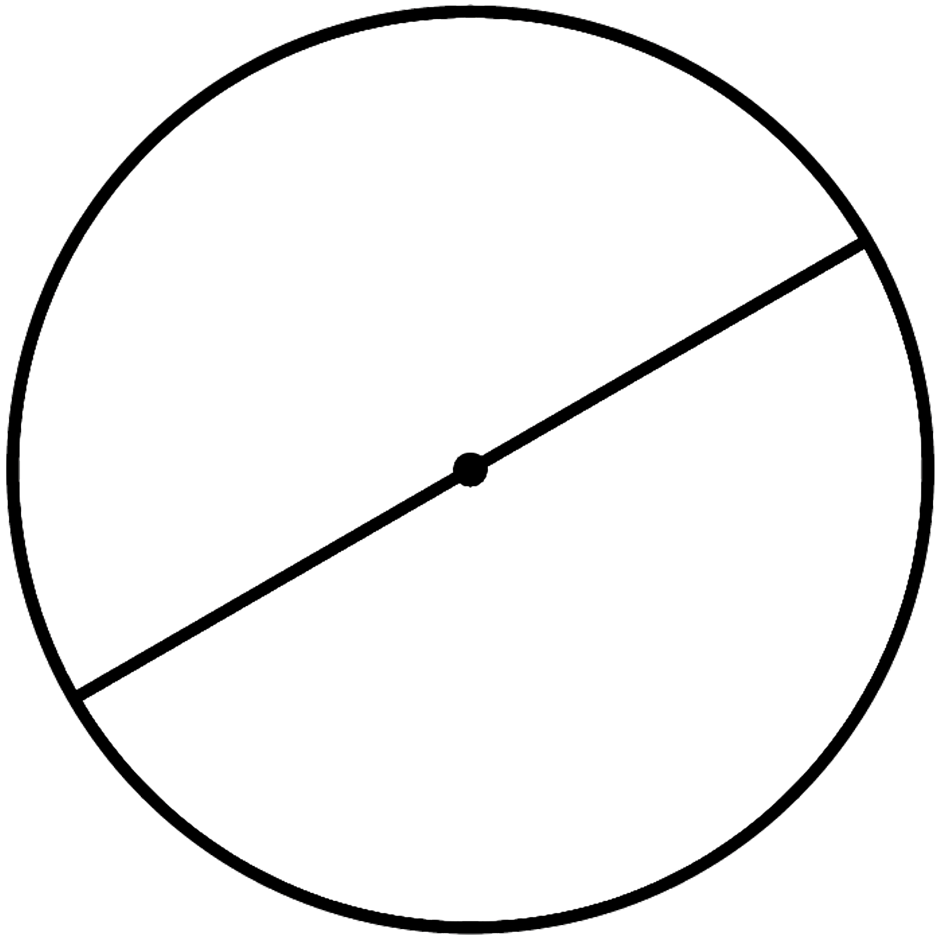
The First Proof

- Thales is the first person to whom a mathematical proof is attributed.
- He’s also called the Father of Science because he used evidence and theories, not mythology, to study and explain the world.
- Aesop had two stories about him.

The First Proof

Theorem. The diameter of any circle divides that circle into two equal halves.

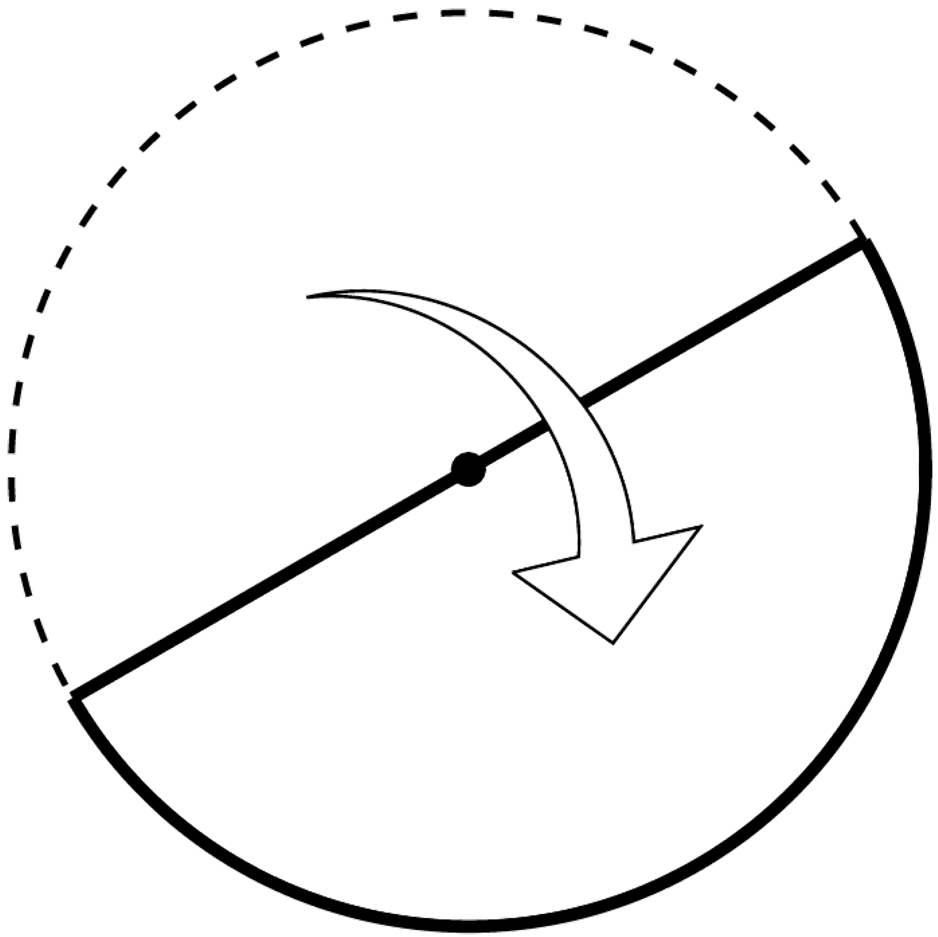
Proof. Suppose you have a circle and diameter



The First Proof

Theorem. The diameter of any circle divides that circle into two equal halves.

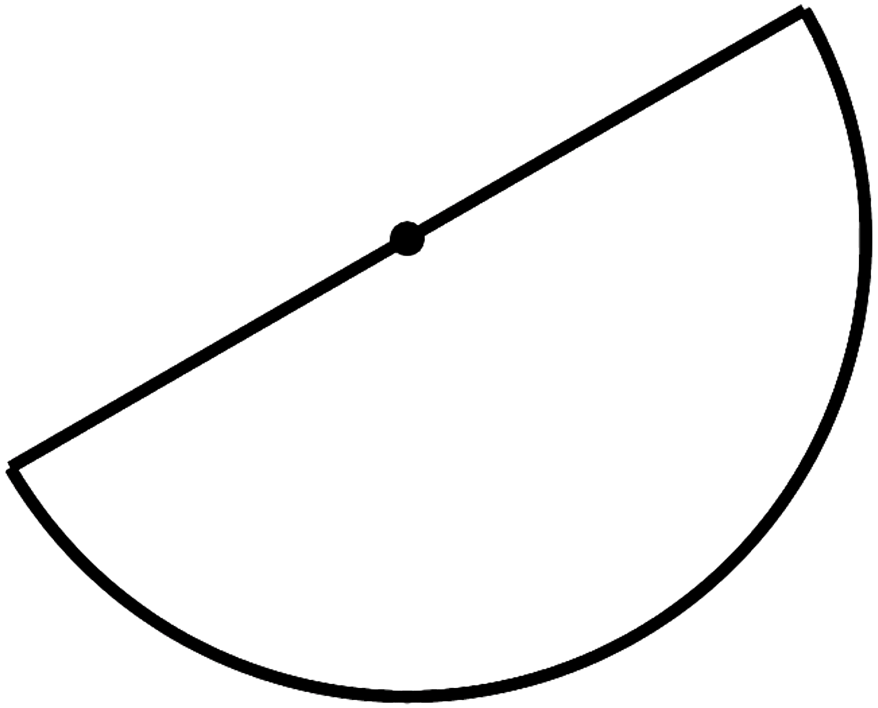
Proof. Take half the circle and flip it over the other half.



The First Proof

Theorem. The diameter of any circle divides that circle into two equal halves.

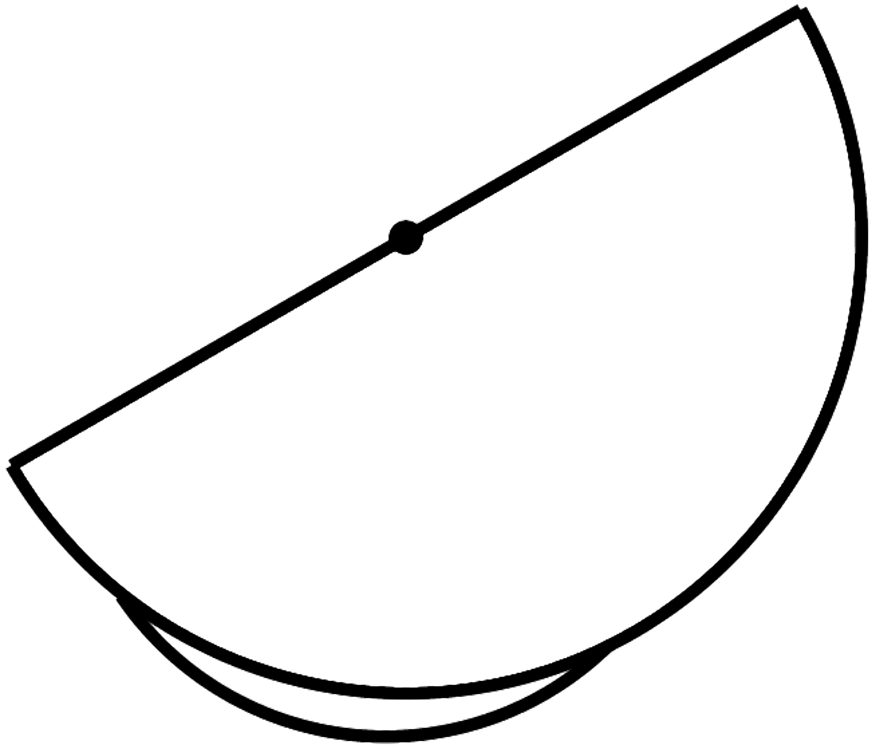
Proof. We want to show that the picture looks like this:



The First Proof

Theorem. The diameter of any circle divides that circle into two equal halves.

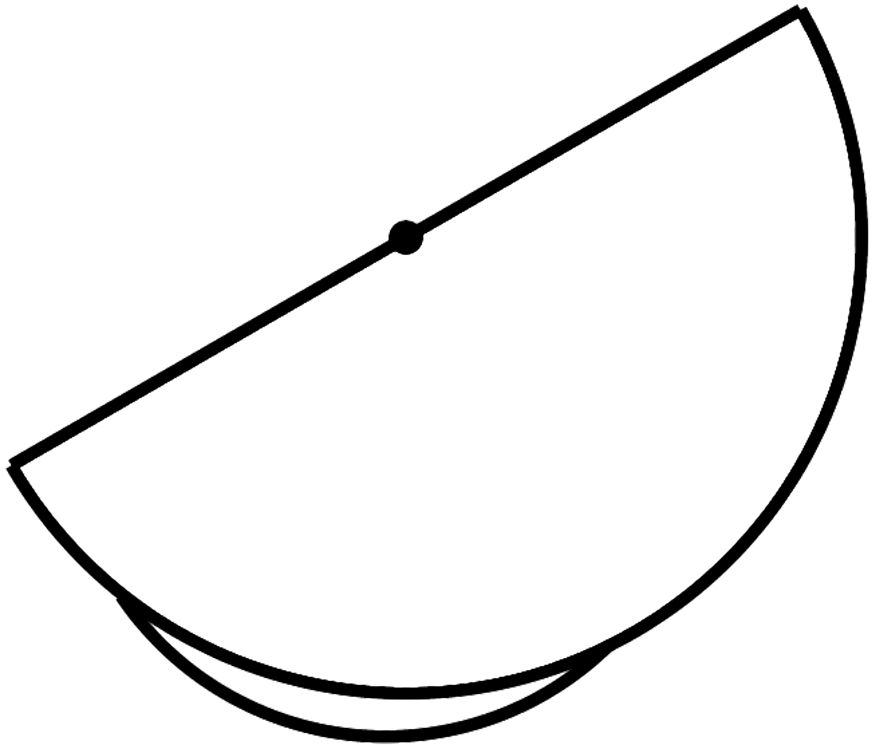
Proof. If it doesn't, then it looks something like this:



The First Proof

Theorem. The diameter of any circle divides that circle into two equal halves.

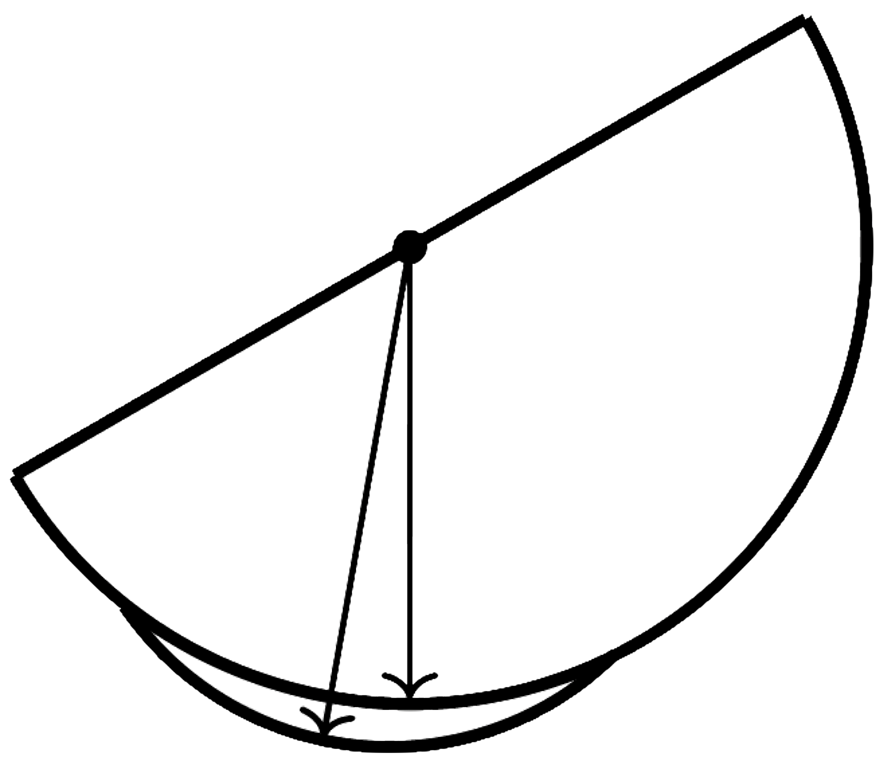
Proof. This gives us a contradiction. Why?



The First Proof

Theorem. The diameter of any circle divides that circle into two equal halves.

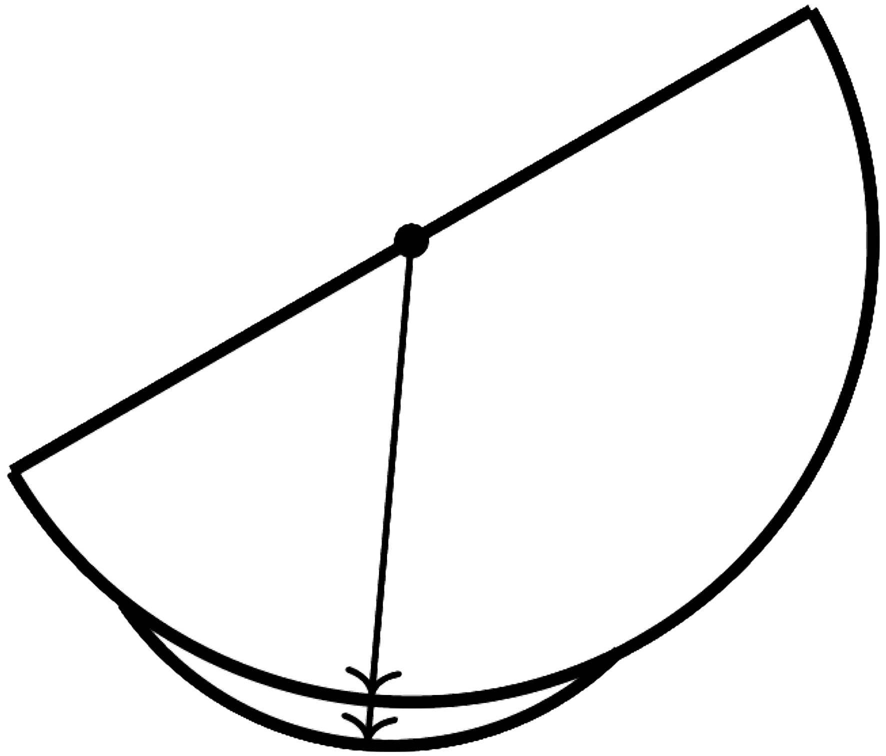
Proof. It’s a contradiction because it produced two separate radii! But a circle has just one radius.



The First Proof

Theorem. The diameter of any circle divides that circle into two equal halves.

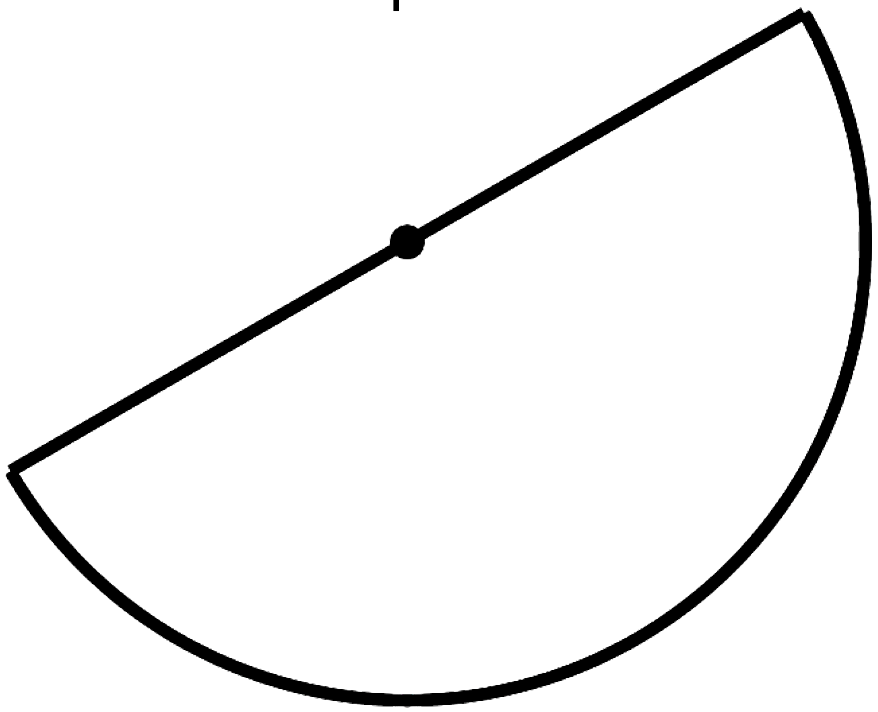
Proof. This fact is even clearer if you line up the two radii that appear.



The First Proof

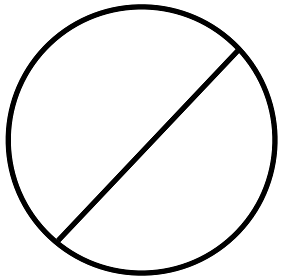
Theorem. The diameter of any circle divides that circle into two equal halves.

Proof. Either way, the conclusion is the same. If the two parts did not perfectly overlap then we would get two radii, which is impossible. So the overlap must be perfect.

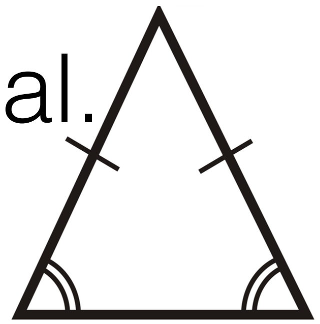


The First Proof

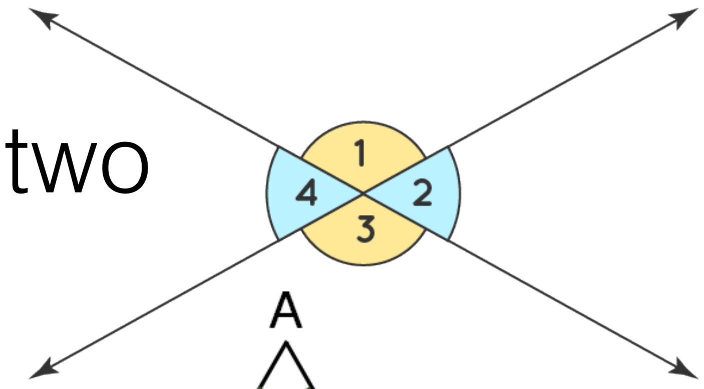
Other things Thales proved:



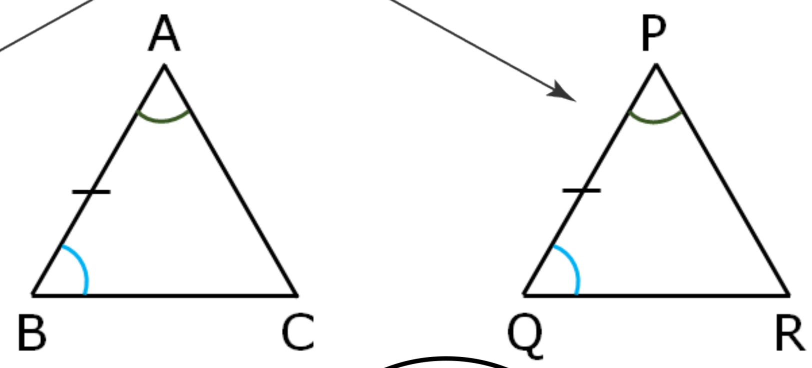
1. A circle is bisected by any diameter.
2. The base angles of an isosceles triangle are equal.



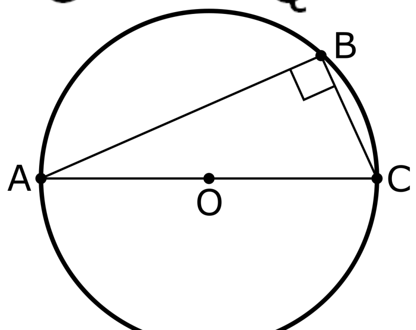
3. The vertical angles formed by two intersecting lines are equal.



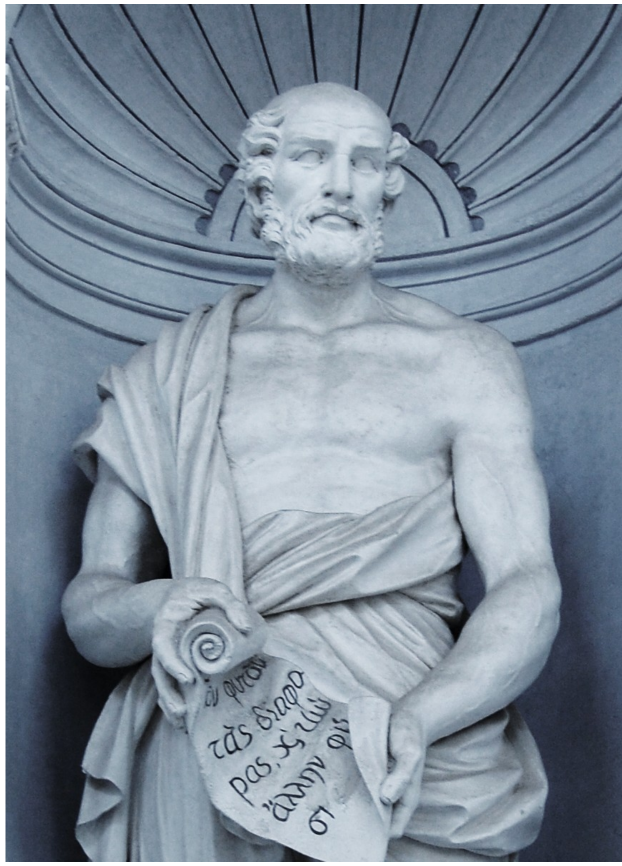
4. The ASA triangle congruence.



5. The angle inscribed in a semicircle is a right angle.



Historial Uncertainty



- Eudemus of Rhodes (~370 BC — ~300 BC) is considered the first historian of science. He also worked closely with Aristotle.
- He wrote *History of Geometry*, a book about all the geometry known to the Greeks. Sadly, no copy of this book exists today.

Historial Uncertainty

- Proclus (412 AD — 485 AD) gave a short sketch of the history of geometry which seems to be based on Eudemus' book *History of Geometry*. He discusses Thales, Pythagoras and others.

- Stories about Pythagoras are often contradictory or clearly false. This casts doubt on the rest.

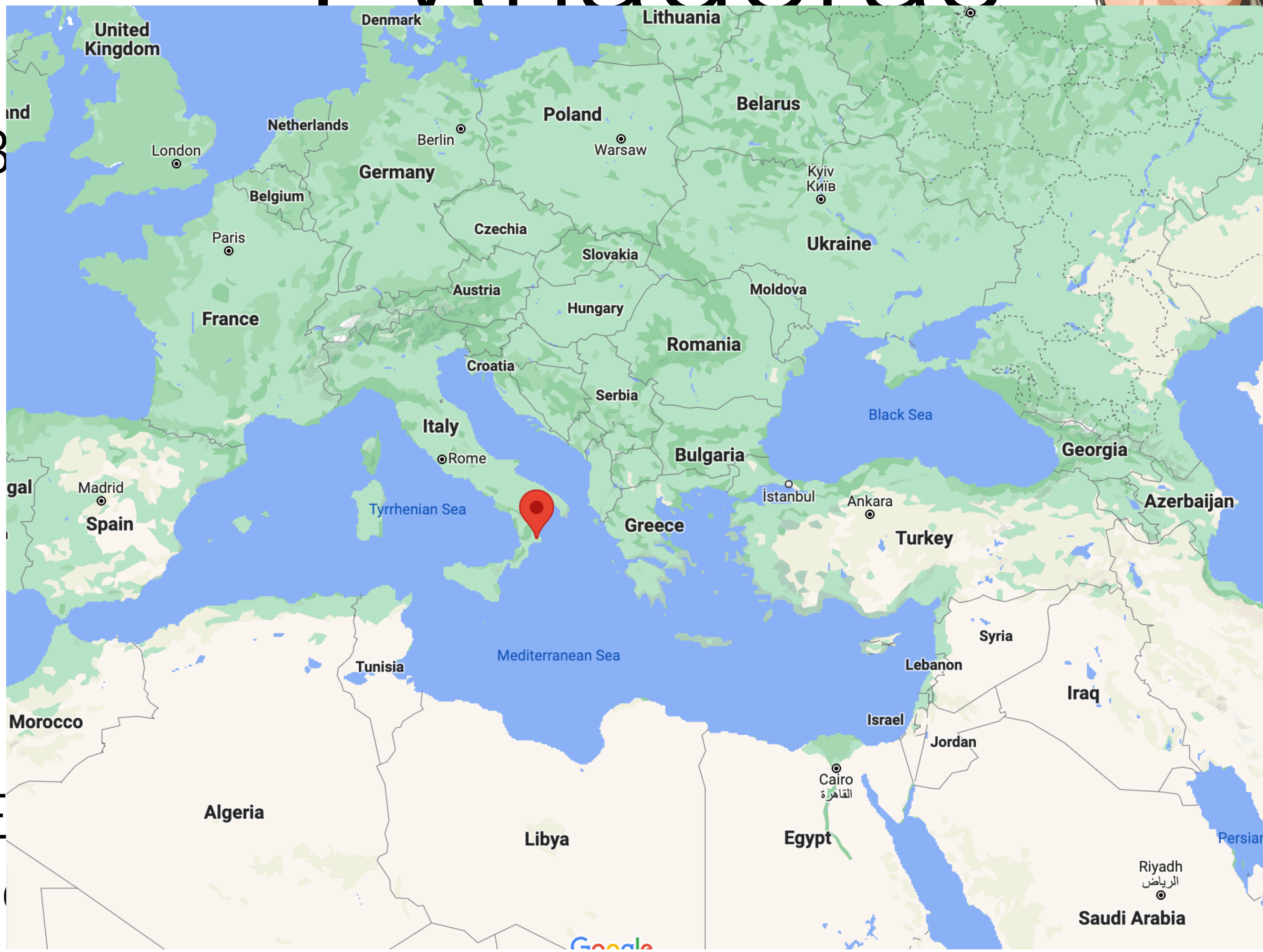
- Yet, from Proclus and a few others, there is (non-conclusive) evidence that Pythagoras made important contributions to math.

Pythagoras

- Born in Samos around 572 BC

Pythagoras

- B



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Pythagorean Brotherhood

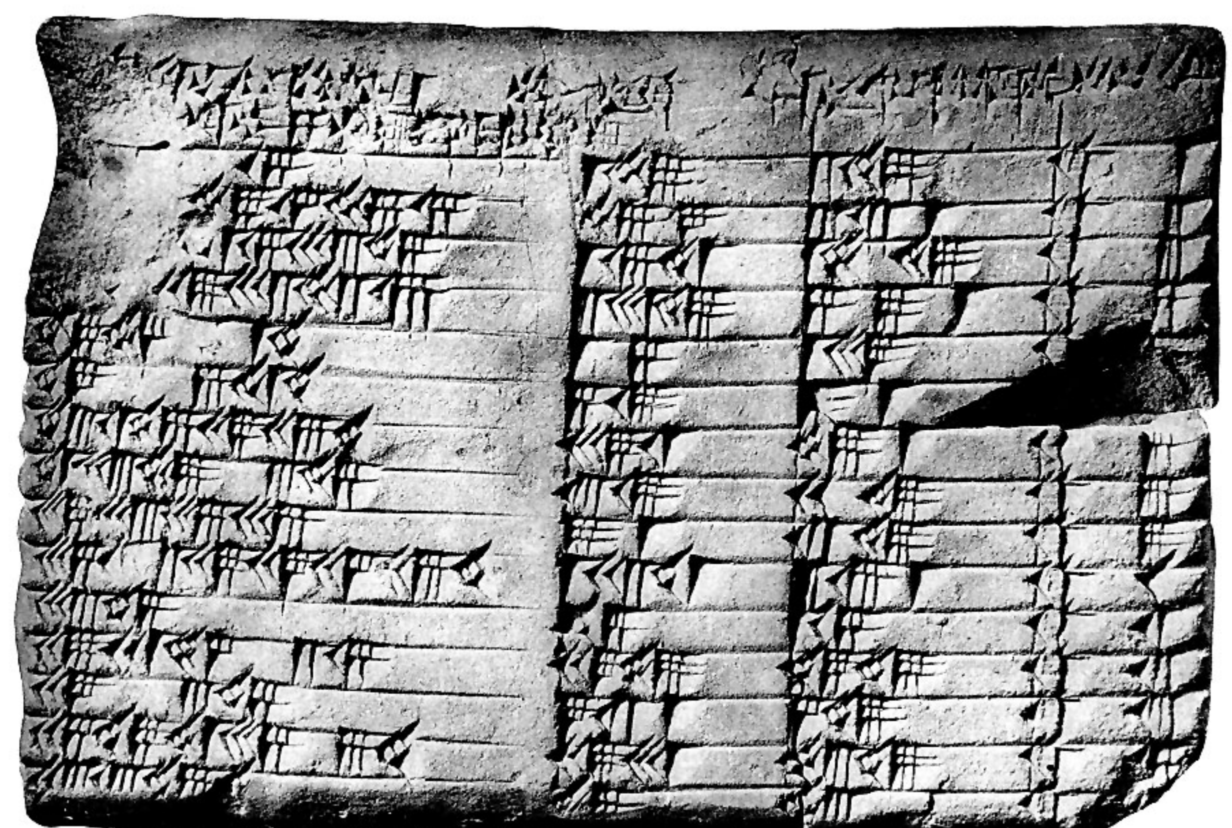
- It was basically a cult, with Pythagoras at the helm. Special diet, exercise, activities, rituals. May have studied numerology and mathematics.

Pythagorean Brotherhood

- Pythagoras (or one of his followers) *may* have proved the Pythagorean theorem.
- If they did, then it was probably a “dissection proof.”
- If it was, then it may have been the proof that I will show you in a moment.

Pythagorean Triples

- Definition: A *Pythagorean triple* is a triple of positive integers (a, b, c) satisfying $a^2 + b^2 = c^2$.
- Example: $3^2 + 4^2 = 5^2$, so $(3,4,5)$ is a Pythagorean triple.
- Recall: 15 PTs appeared in Plimpton 322 (~1800 BC).
- Nearly all ancient cultures found some Pythagorean triples.

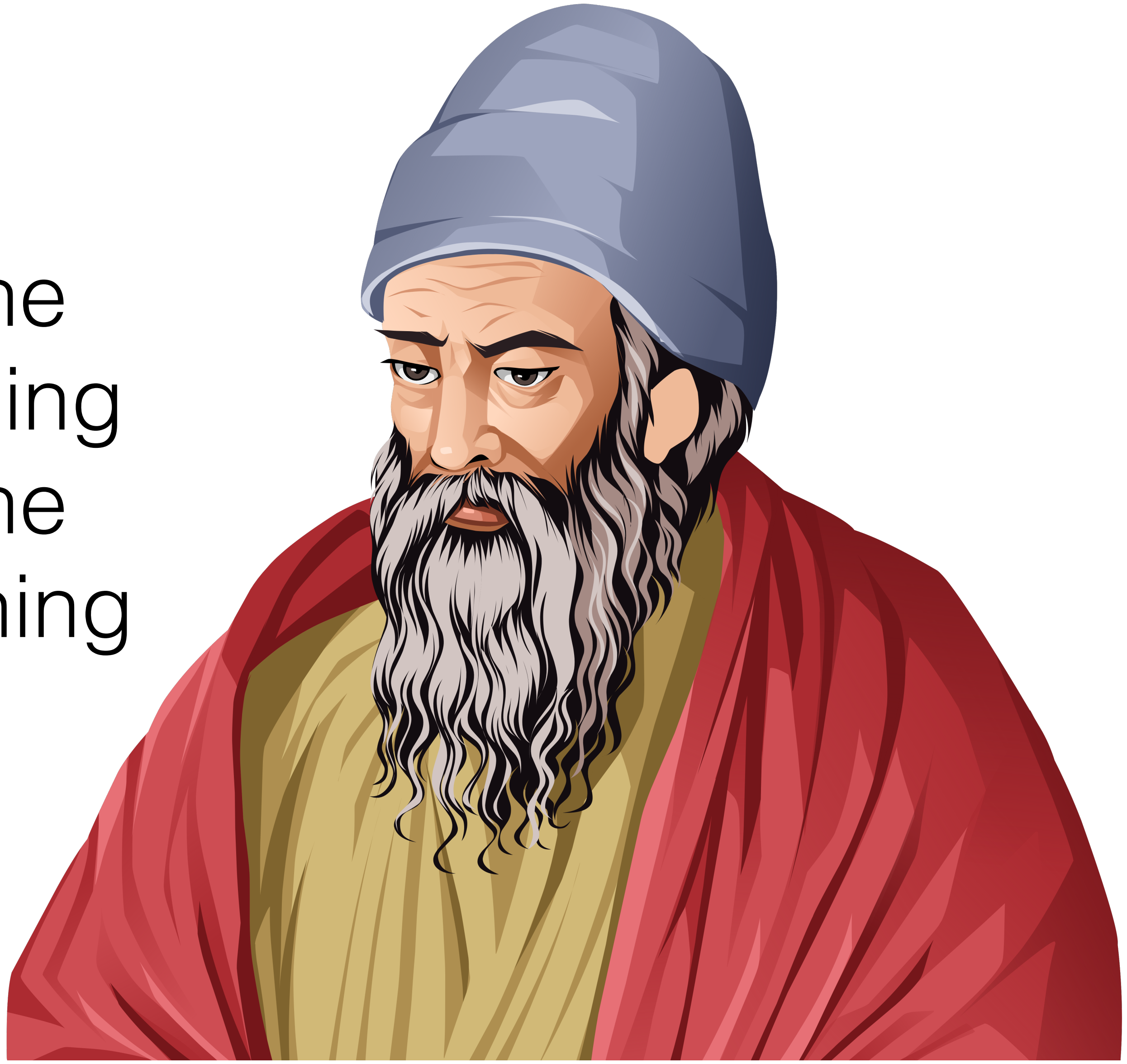


Pythagorean Theorem

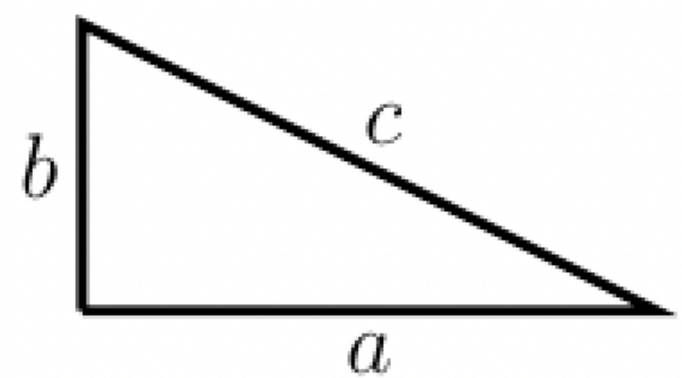
- Warm-up: State the Pythagorean theorem.

- Euclid:

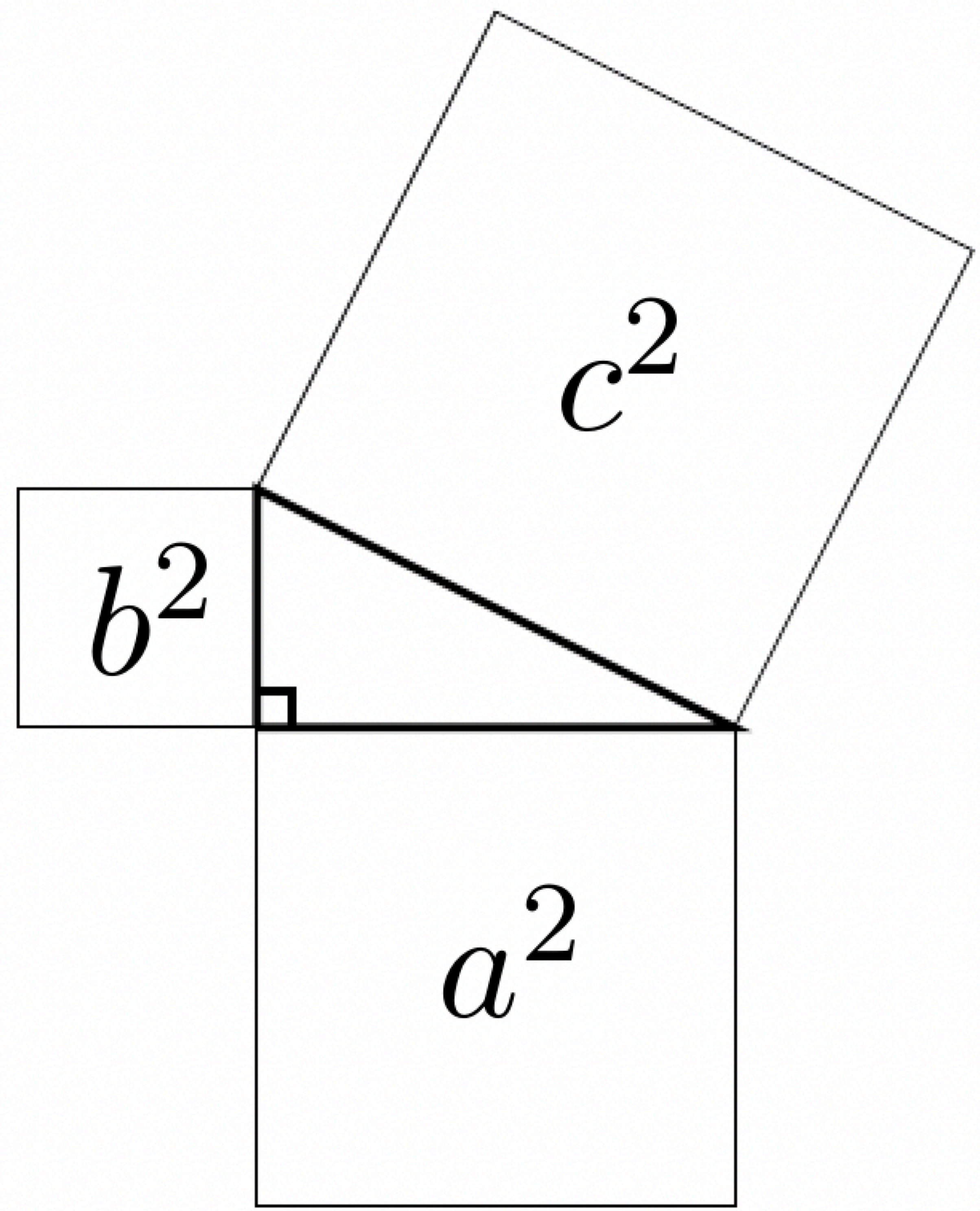
  “In right-angled triangles, the square on the side subtending the right angle is equal to the squares on the side containing the right angle.”



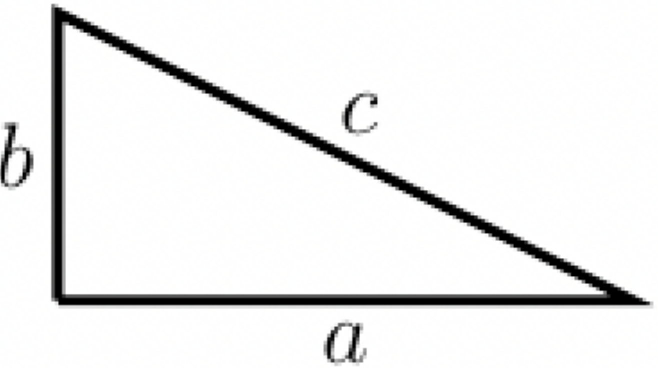
Pythagorean Theorem

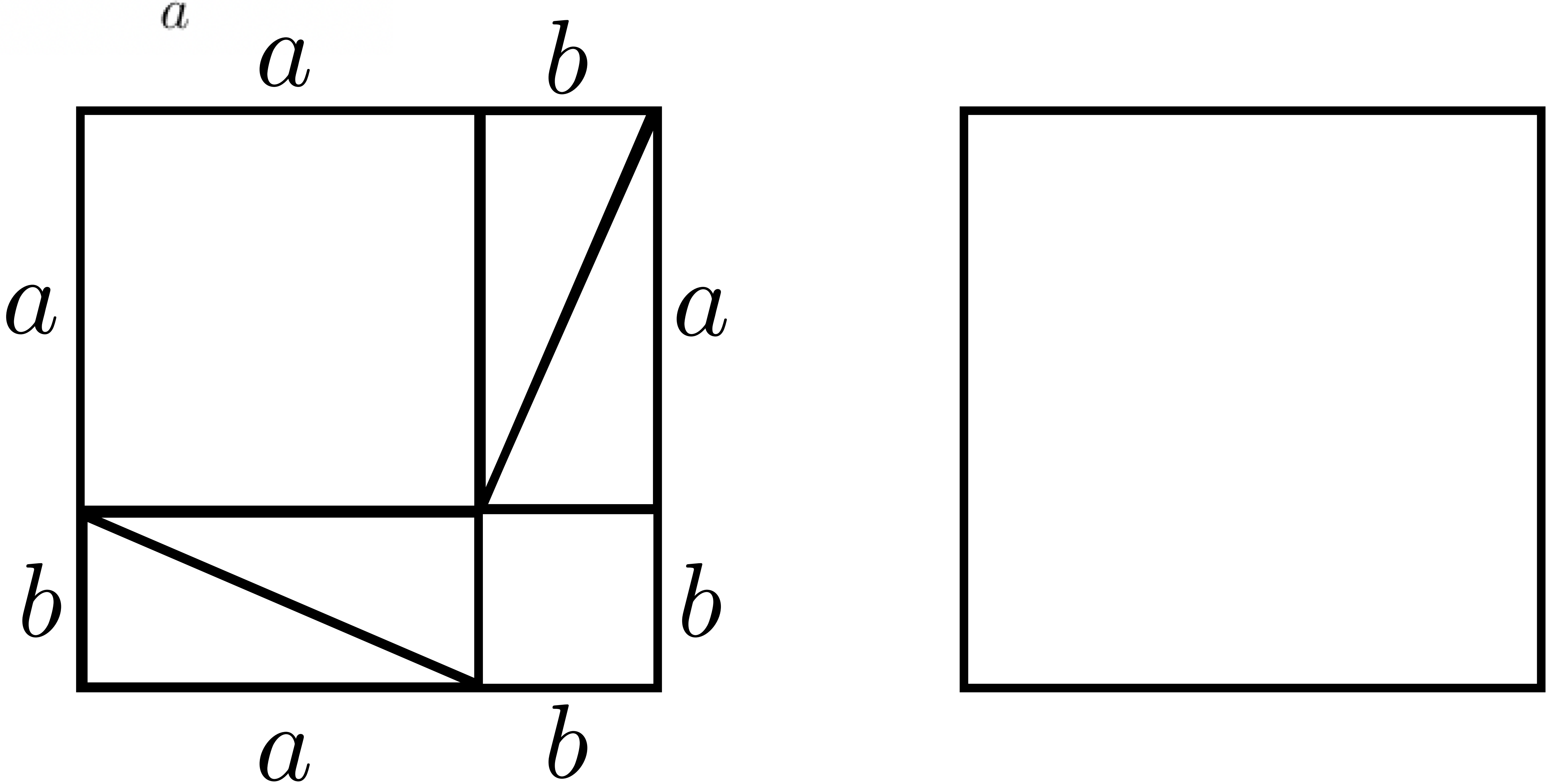


$$a^2 + b^2 = c^2$$

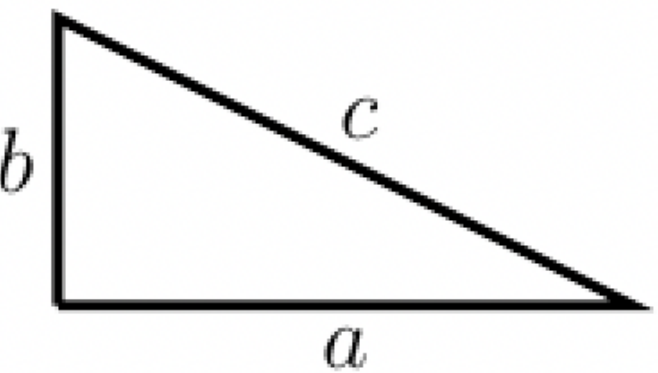


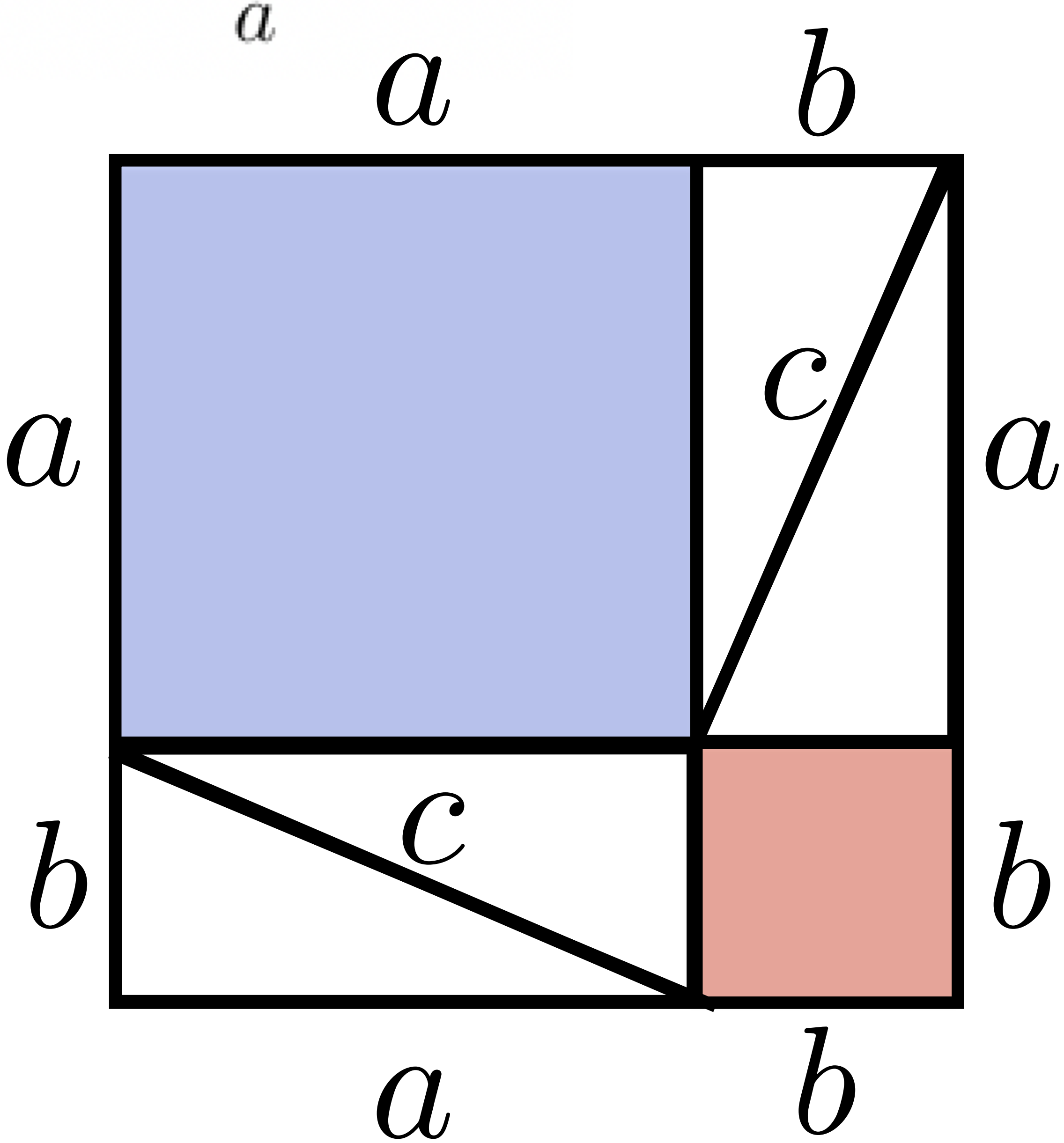
How They *Might* Have Proved It

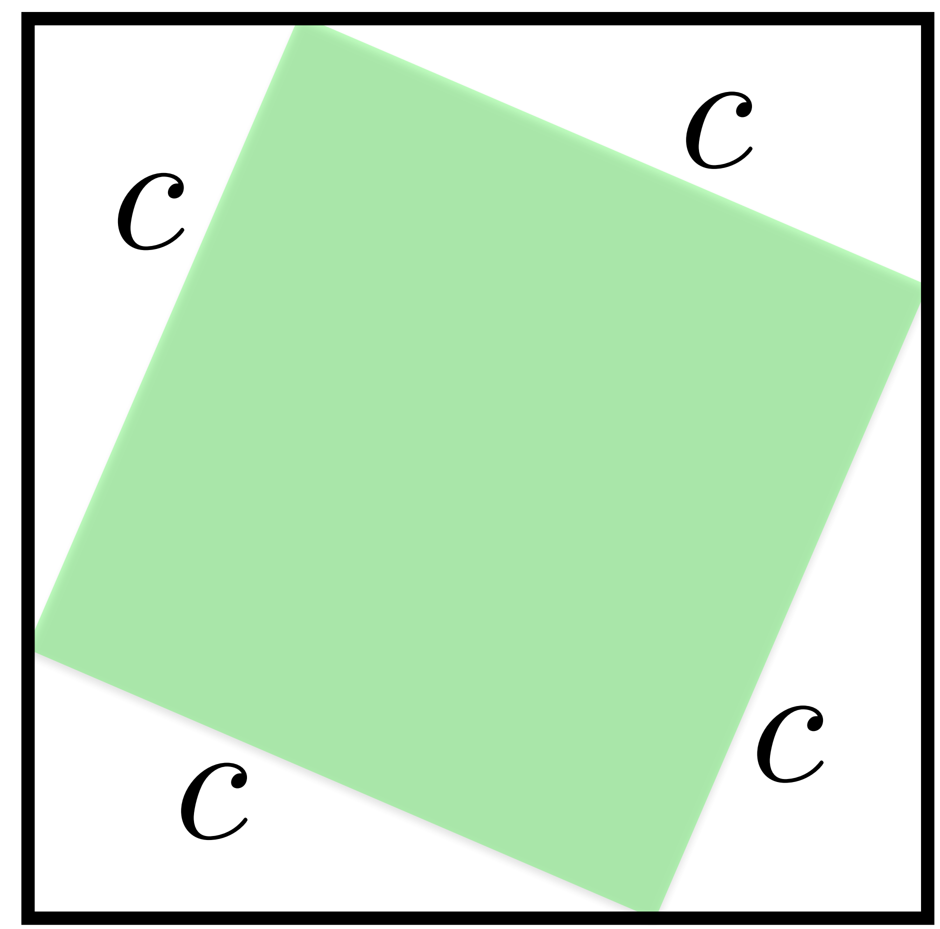




How They *Might* Have Proved It

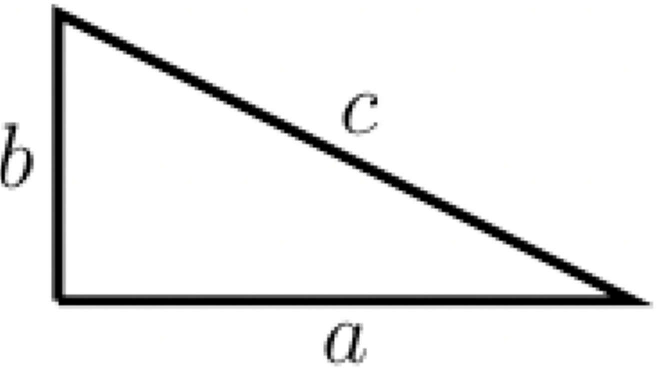


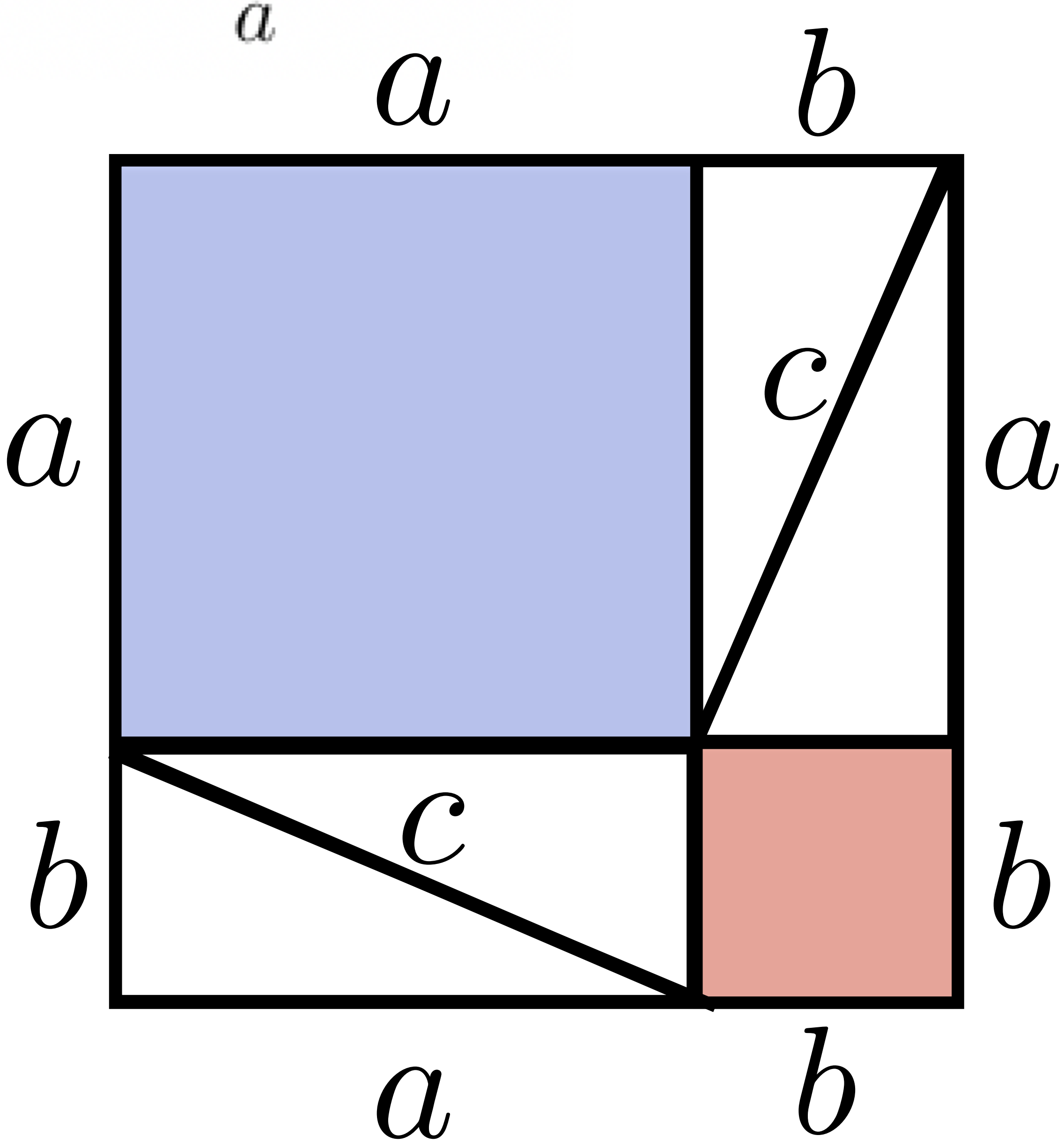


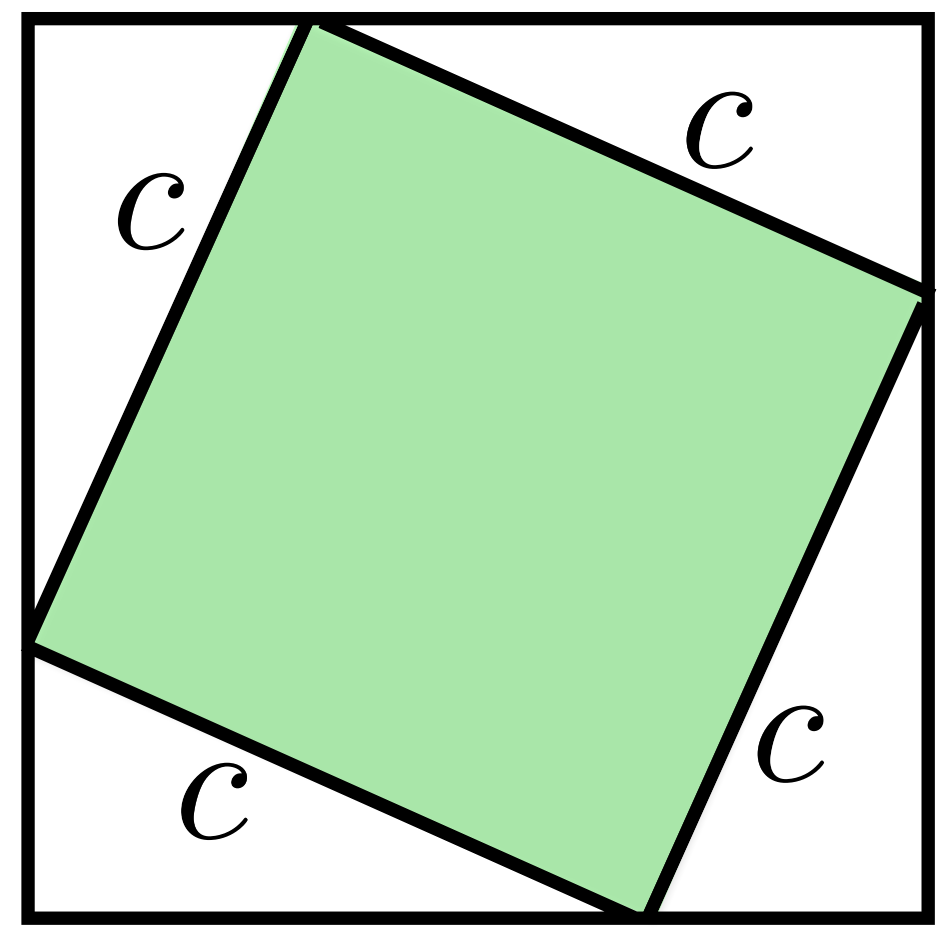


$a^2 + b^2$ + ~~4 triangles~~ $=$ c^2 + ~~4 triangles~~

How They *Might* Have Proved It







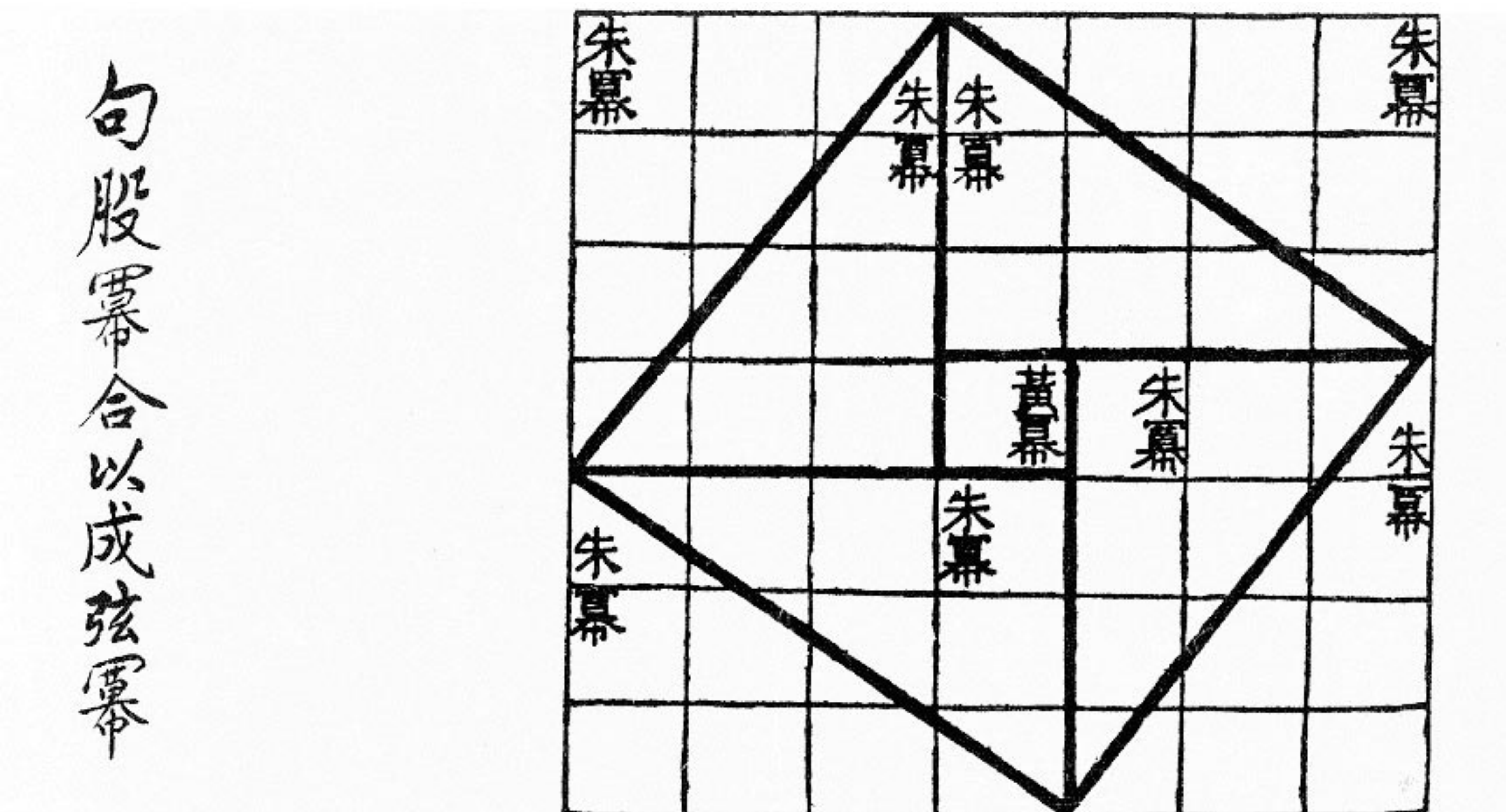
$$a^2 + b^2 = c^2$$

A *Possible* Proof

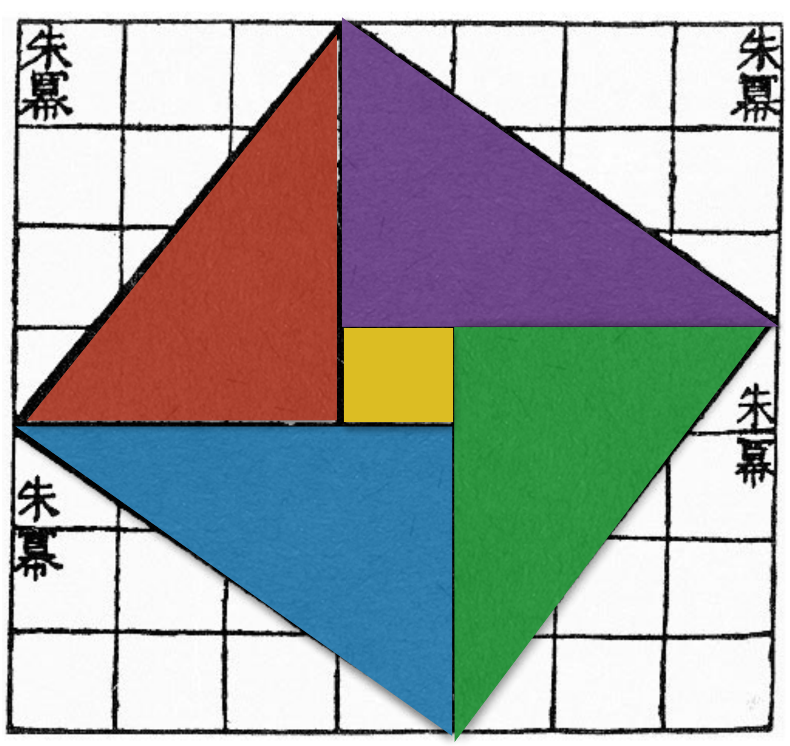
- One of the oldest math texts in China is *Zhoubi Suanjing*. It was written around the time of Pythagoras.
- The book uses the Pythagorean theorem many times and alludes to a diagram.
- A diagram was added to the text in the 3rd century. Unclear if it was the one intended by the original author.
- Time to Think Like A Math Historian!

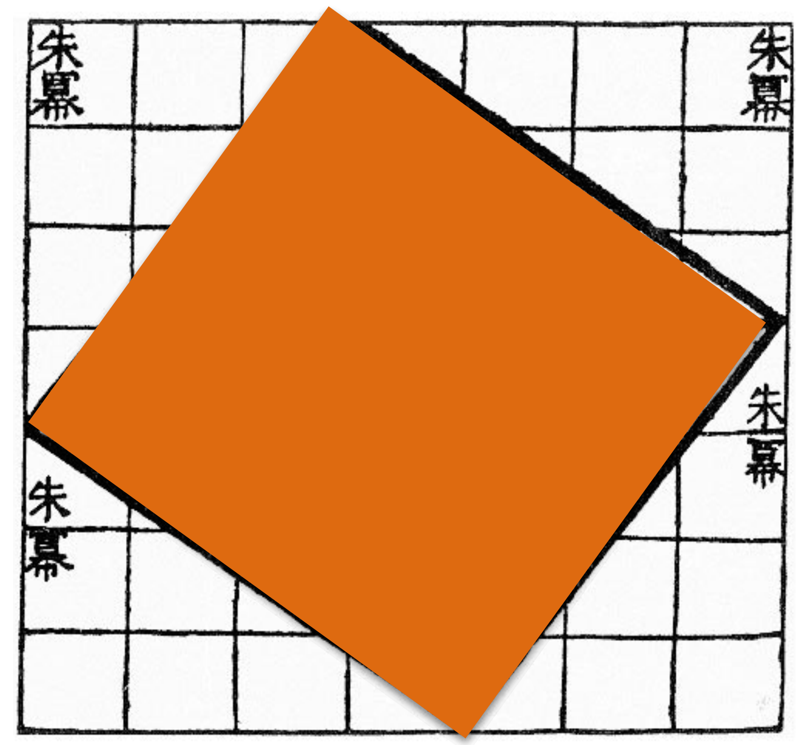
Think Like A Math Historian

Think Like A Math Historian



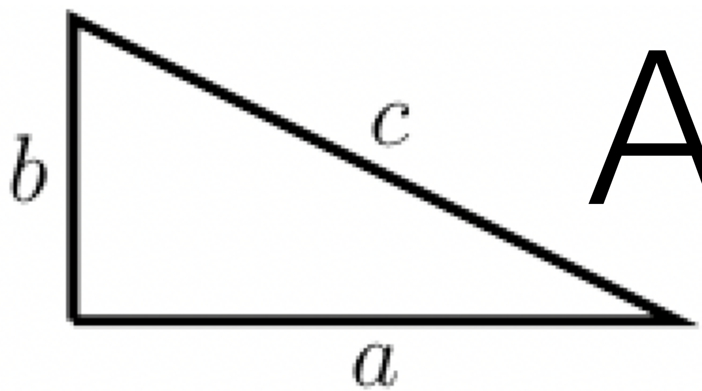
First *Possible* Proof



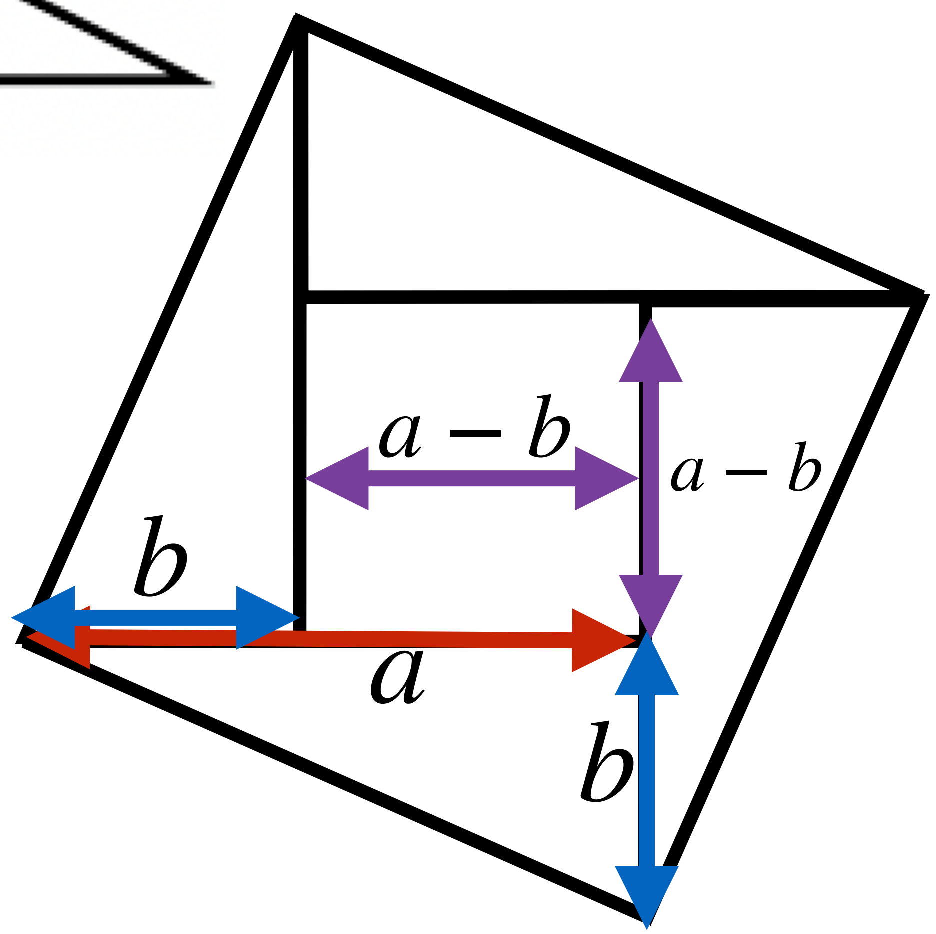


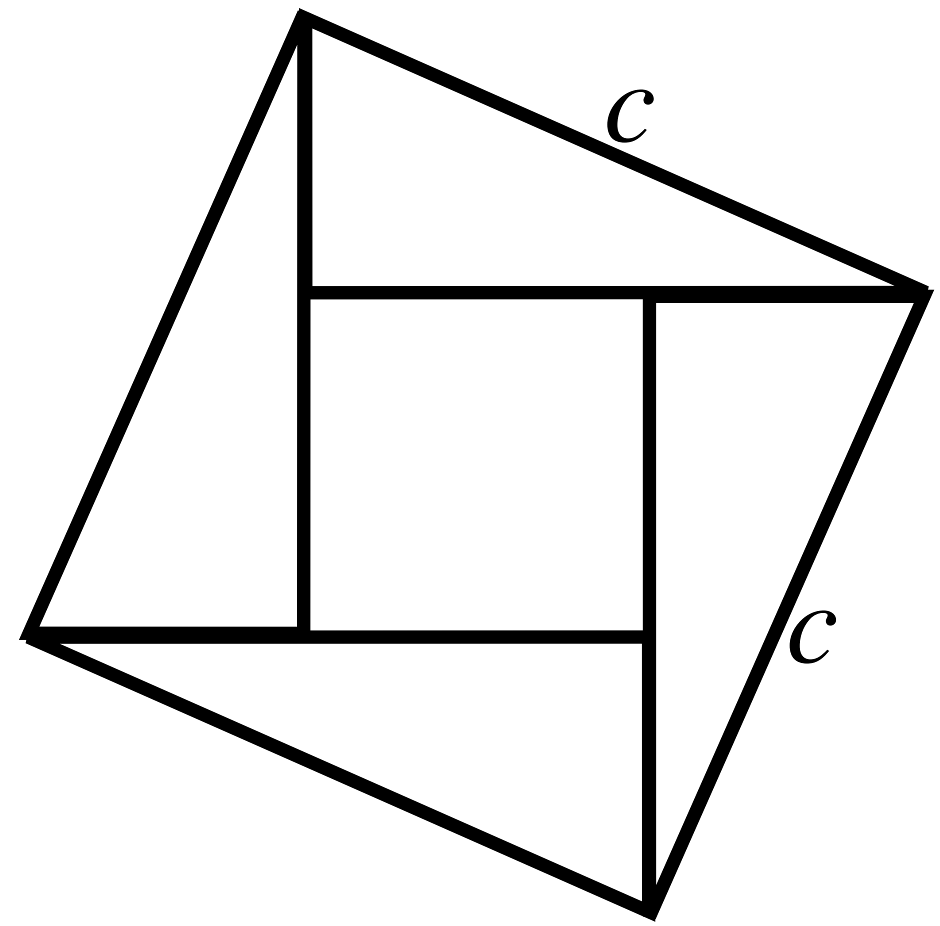
4 triangles + little square = Big square

$$4 \cdot \frac{1}{2}(3 \cdot 4) \quad + \quad 1 \cdot 1 \quad = \quad 5^2$$



A *Possible* Proof





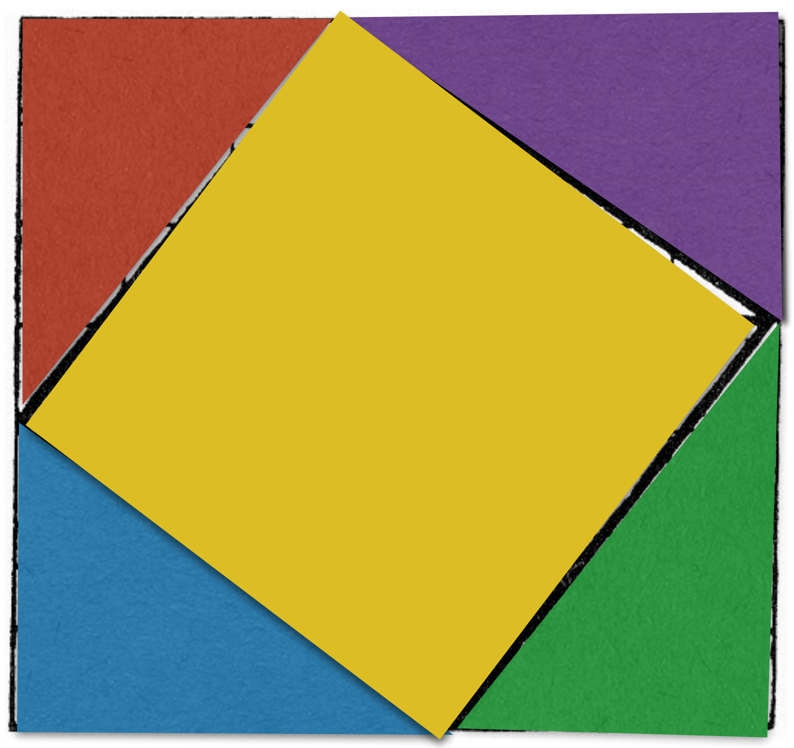
4 triangles + little square = Big square

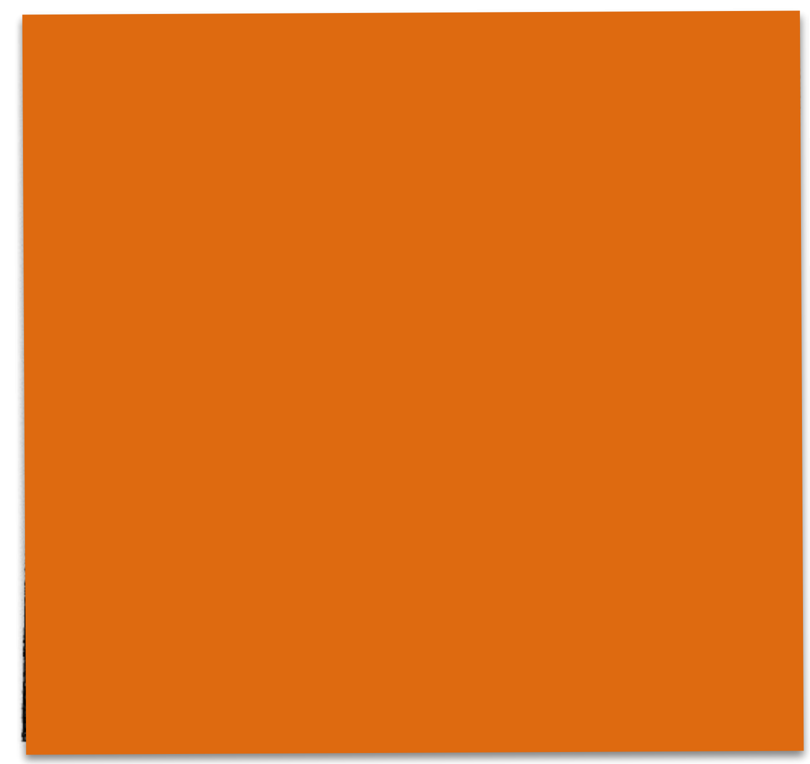
$$4 \cdot \left(\frac{1}{2} a \cdot b\right) + (a-b)^2 = c^2$$

$$\cancel{2ab} + a^2 + b^2 - \cancel{2ab} = c^2$$

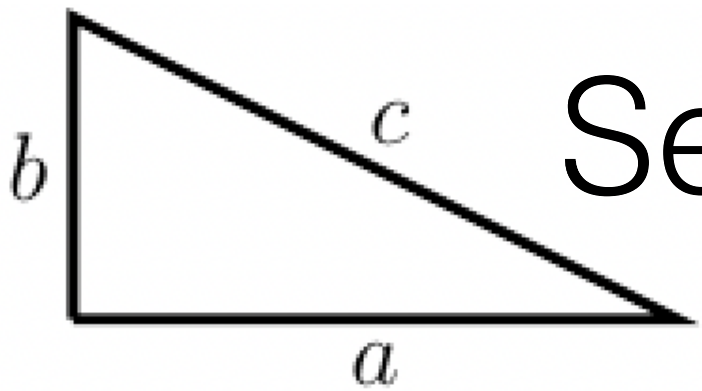
$$\boxed{a^2 + b^2 = c^2}$$

Second *Possible* Proof

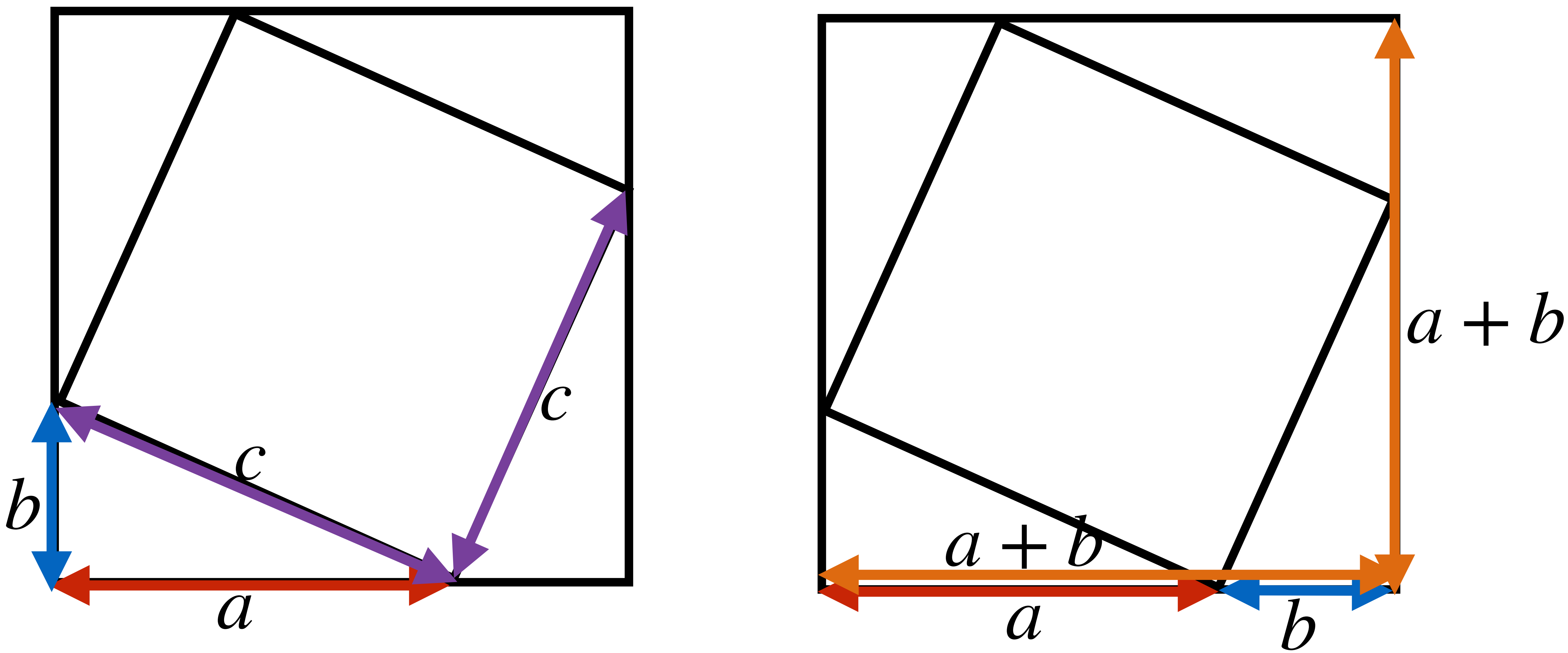




4 triangles + middle square = Whole square



Second *Possible* Proof



4 triangles + middle square = Whole square

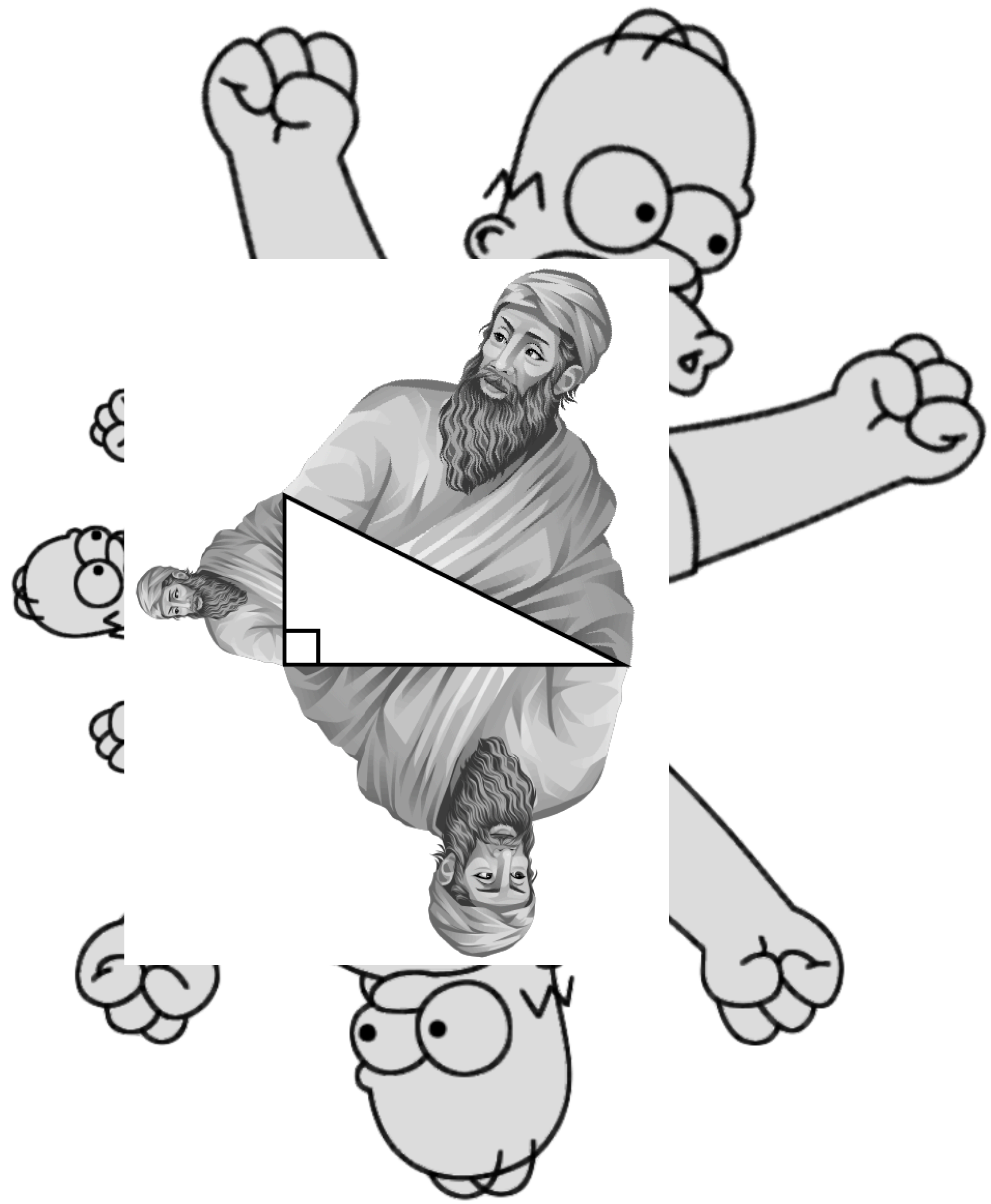
$$4\cdot\left(\frac{1}{2}a\cdot b\right) + c^2 = (a+b)^2$$

$$\cancel{2ab} + c^2 = a^2 + b^2 + \cancel{2ab}$$

$$\boxed{a^2 + b^2 = c^2}$$

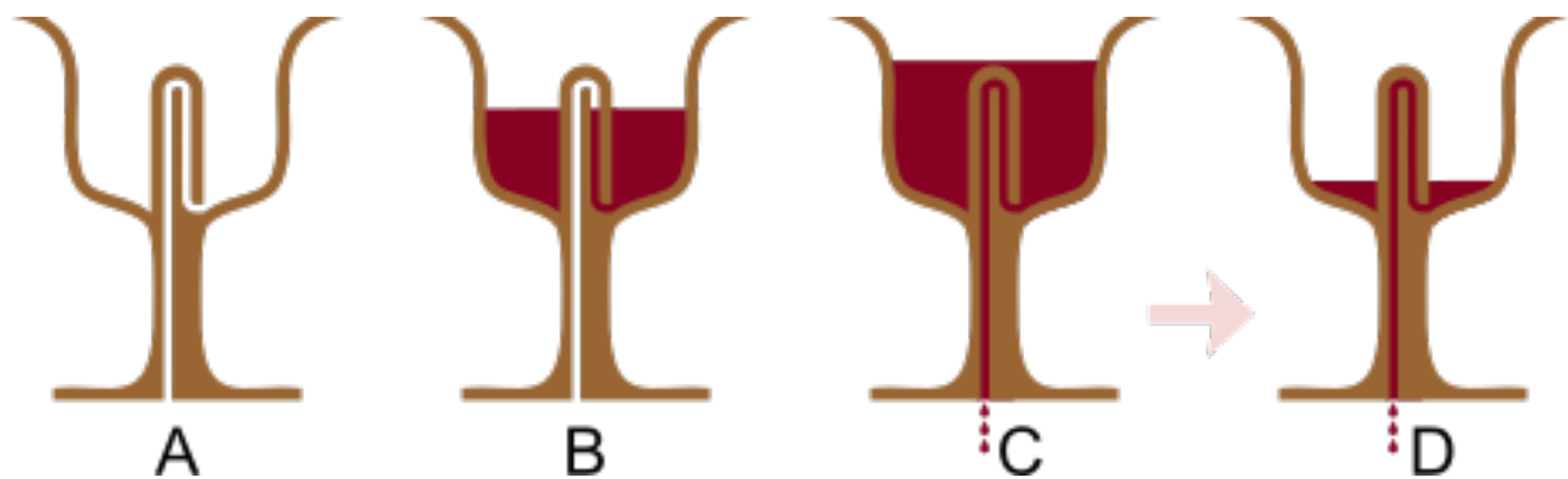
Generalized Pythagorean

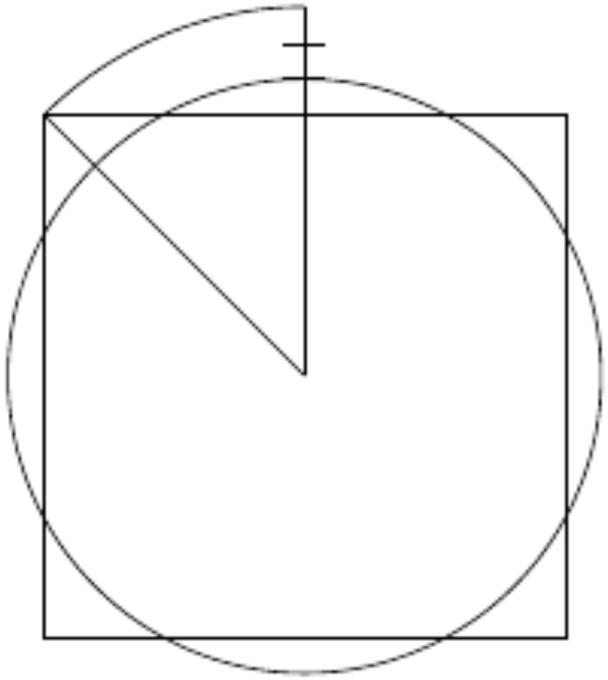
If you attach *any* shape to the sides of a right triangle, the area inside the largest one equals the sum of the areas inside the smaller two.

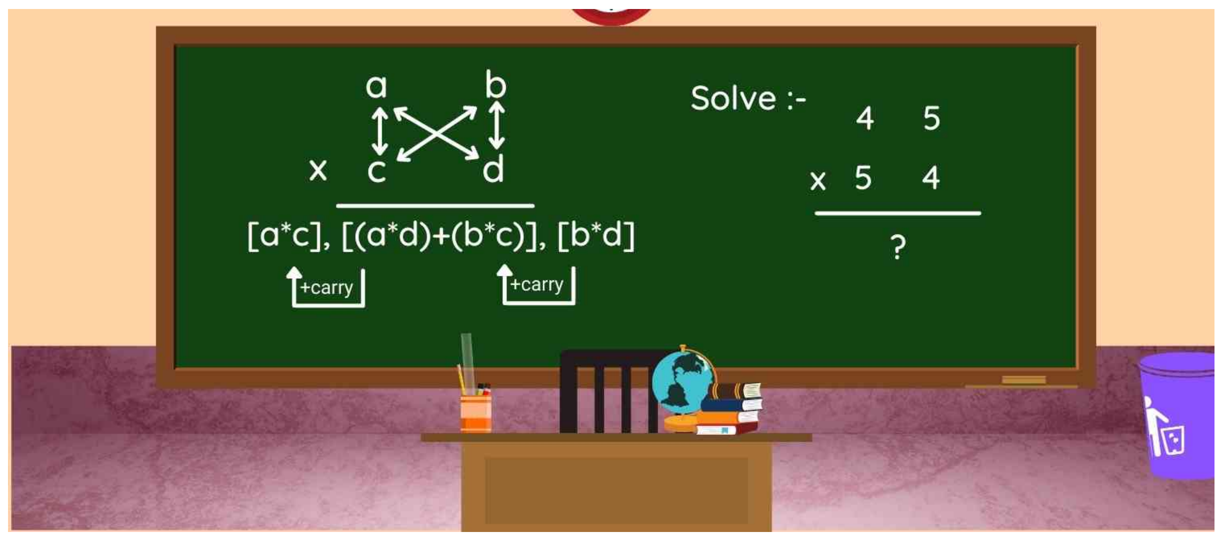


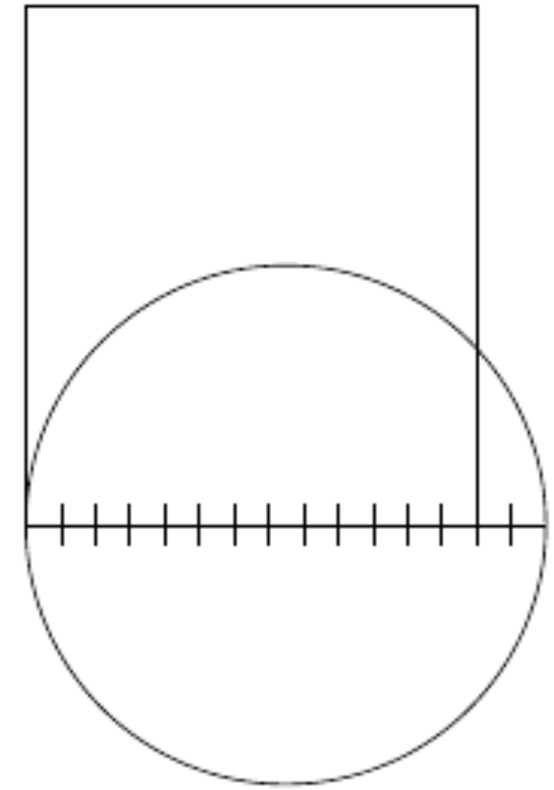
Pythagorean Cup

- A practical joke device whose invention is credited to Pythagoras.
- Also called a “justice cup”

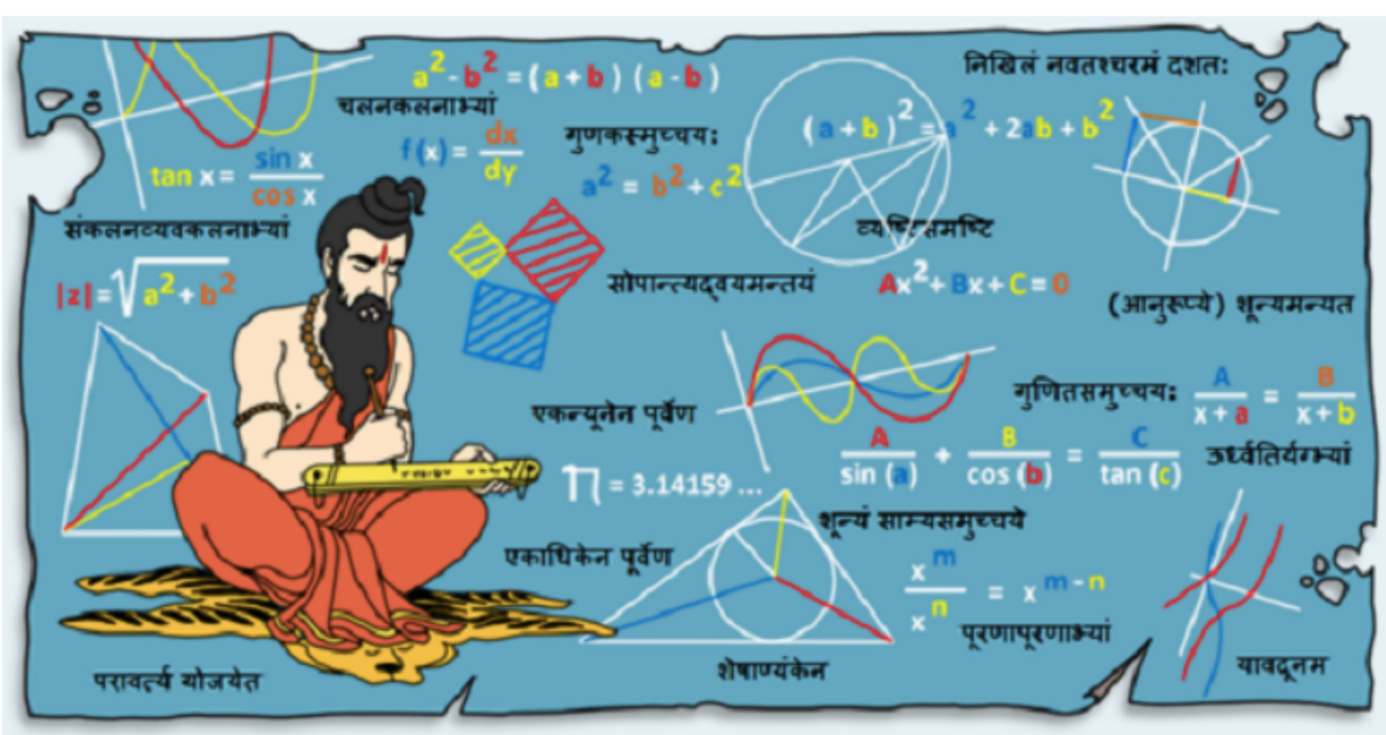


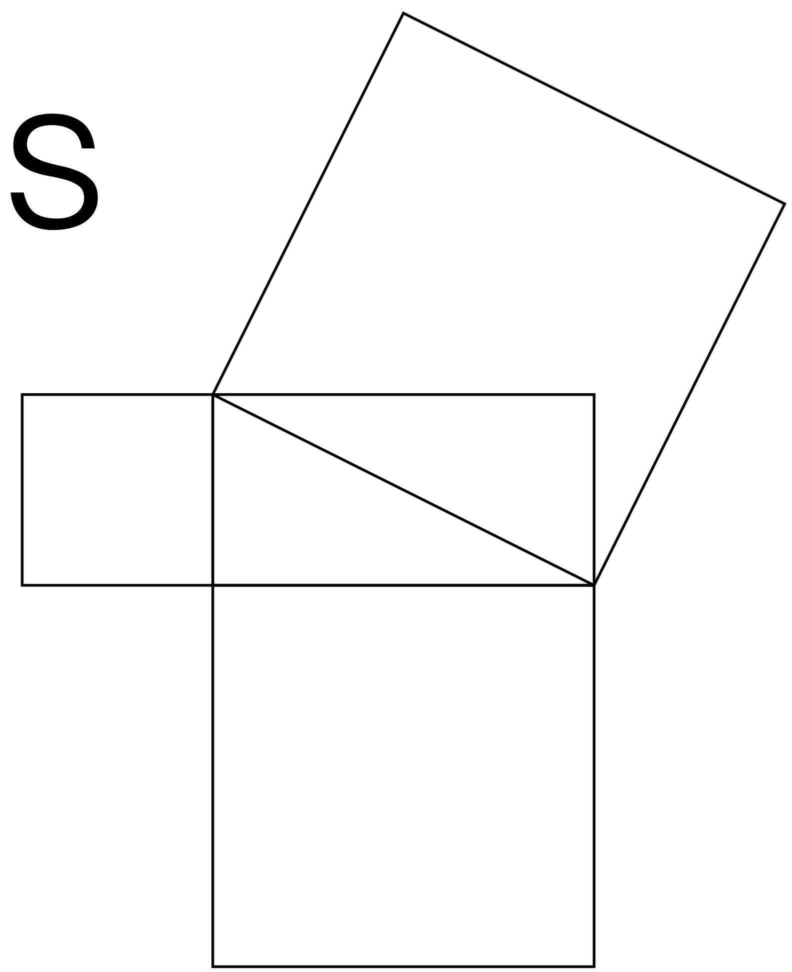






Ancient Indian Mathematics





Ancient India

- Math in ancient India can be divided into eras.
- First era (3000BC-1500BC): Earliest civilizations developed, archeological evidence shows they built cities, houses from brick, grid-organized streets, sewage system, standardized weights and measures. Certainly required some practical math.

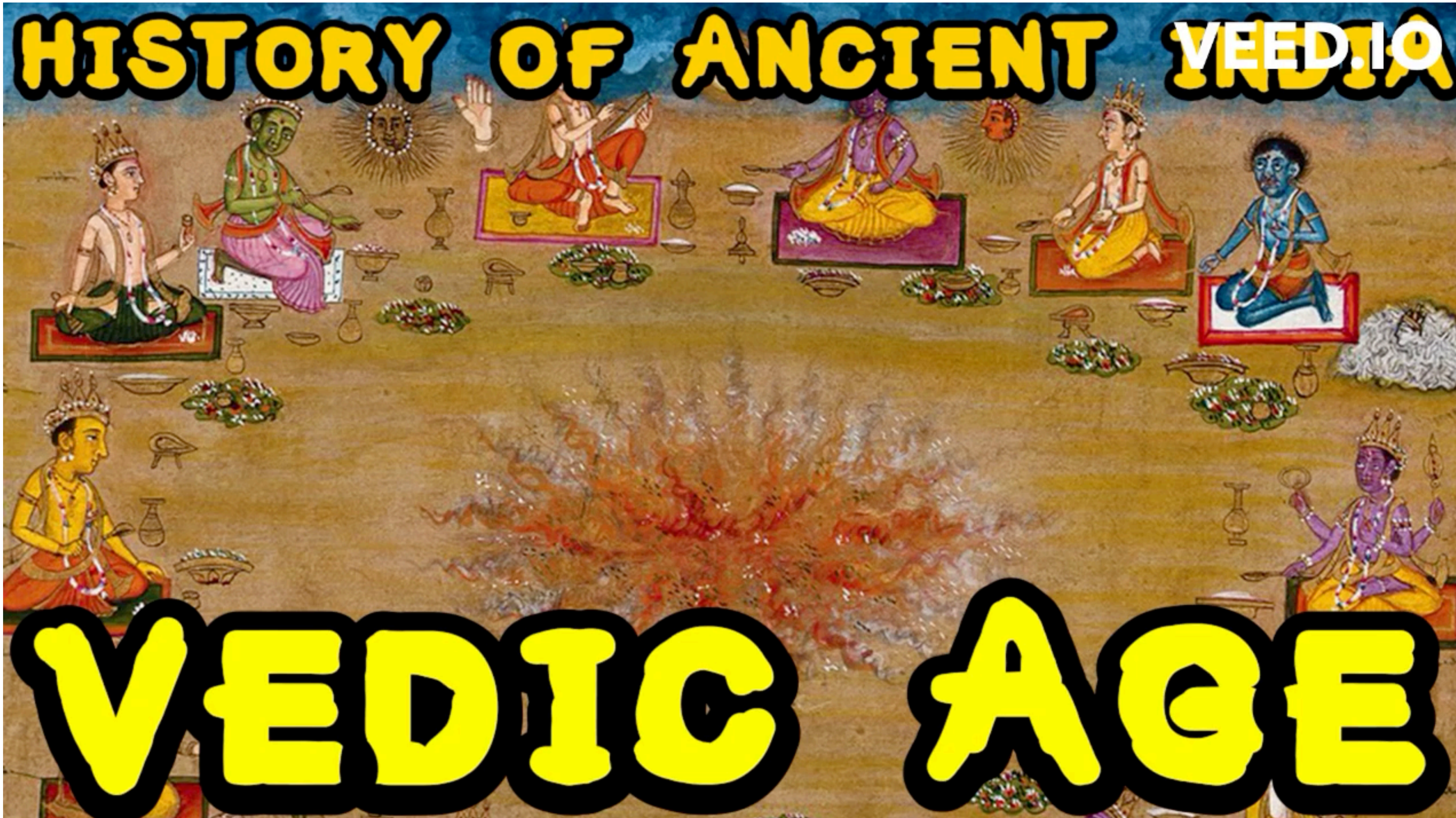
Ancient India

- Second era (1500BC-500BC): Called the *Vedic period.* The Vedic people migrated from modern-day Iran to the Indian subcontinent around 1500 BC.

- "Vedic" refers to the Vedas—the oldest religious texts of Hinduism. The Vedas contain appendices called *sulbasutras*, which contain geometric instructions for how to build fire-altars.

- These tell us some of the geometry they knew.

Ancient India



Ancient India

- The Vedas contain appendices called *sulbasutras*, which contain geometric instructions for how to build fire-altars, which utilized the Pythagorean theorem well.

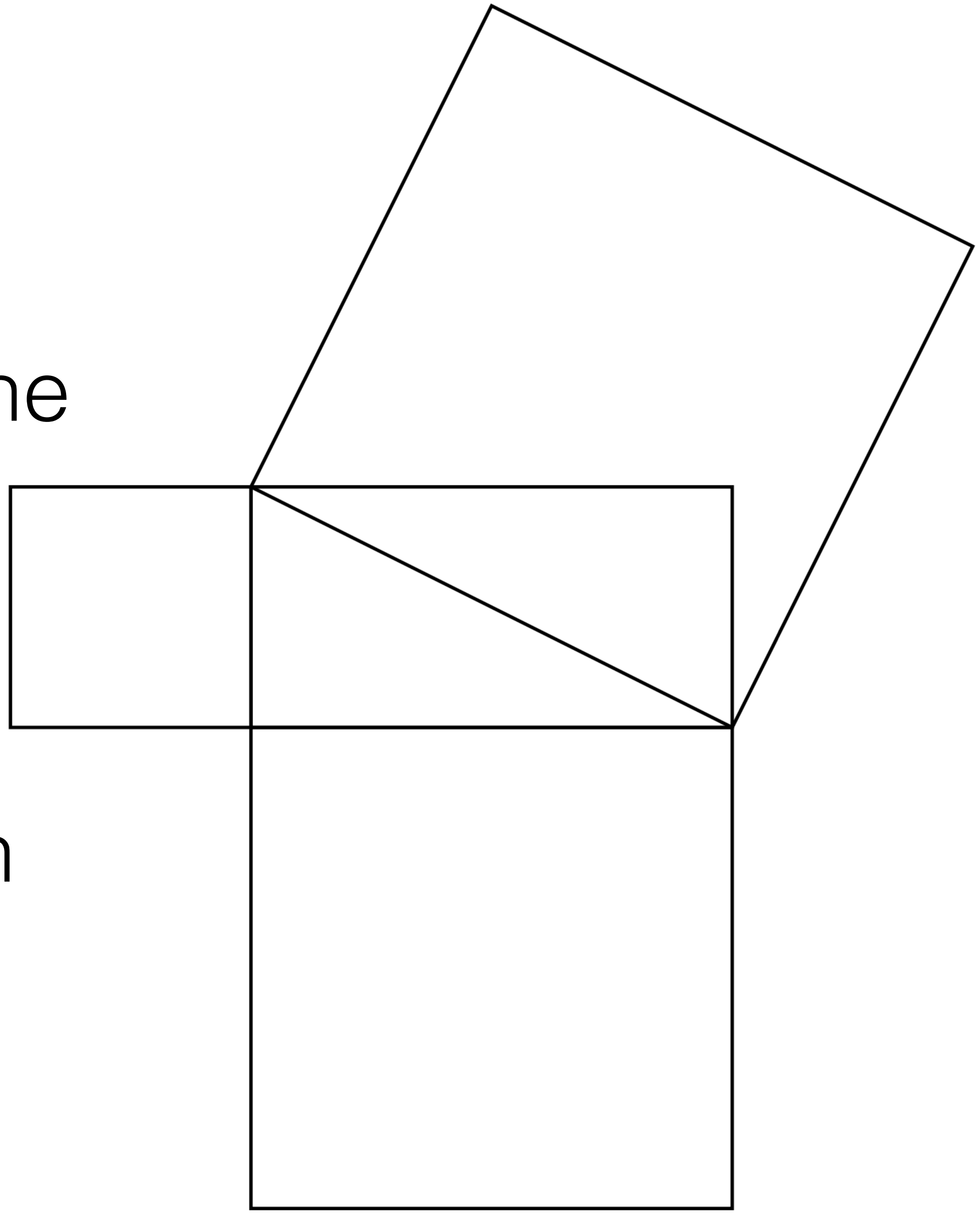
- These tell us some of the geometry they knew.

- There is no direct evidence that they proved the Pythagorean theorem, but they used it so well that they may have.

Ancient India

- Their version of the Pythagorean theorem:

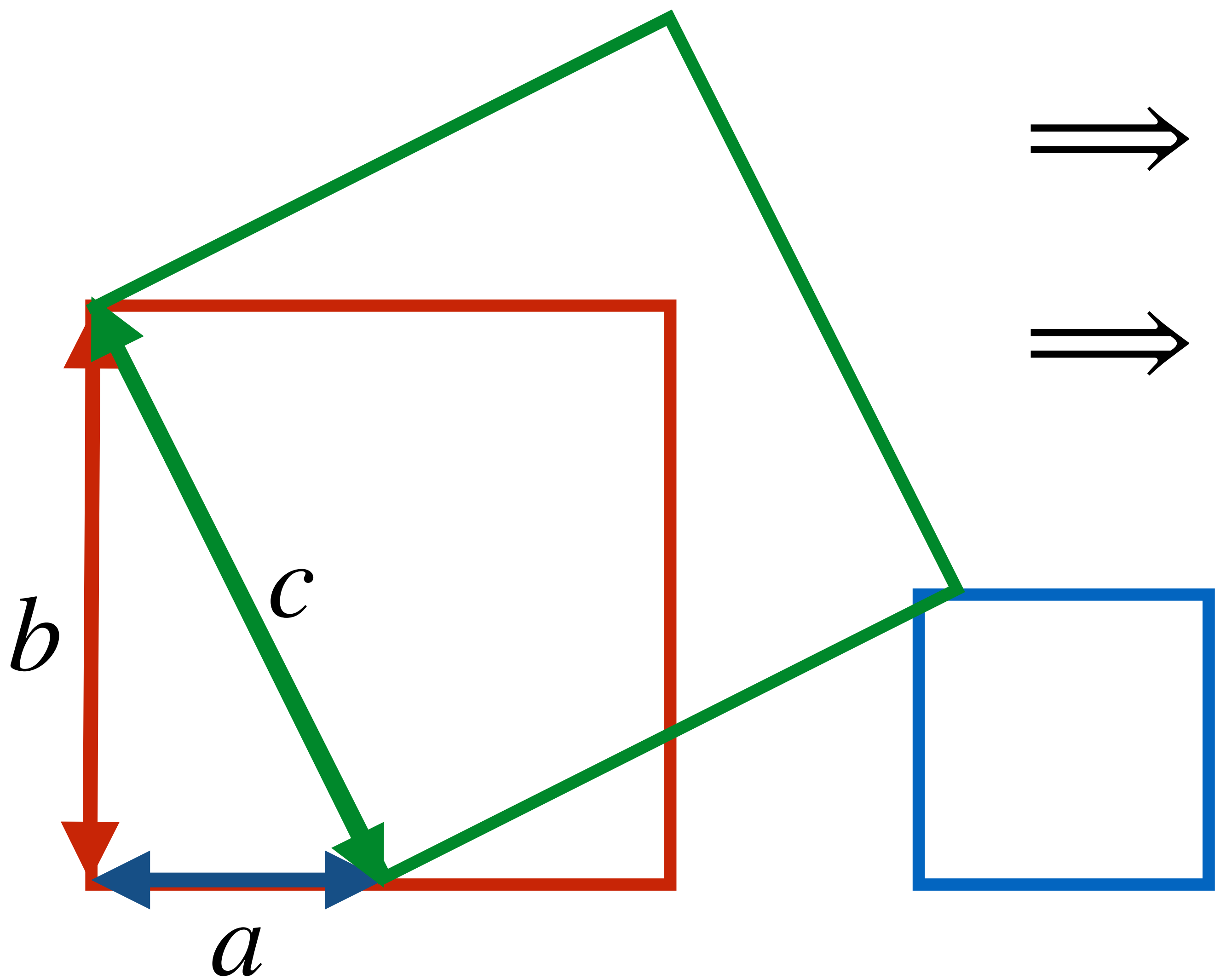
“The rope of the diagonal of a rectangle makes an [area] which the vertical and horizontal sides make together.”



- Three examples of using this in their *sulbasutras*:

Ancient India

- **1. Squaring a pair of squares.** Meaning: Given a pair of squares, create a third square whose area is equal to the sum of the other two.



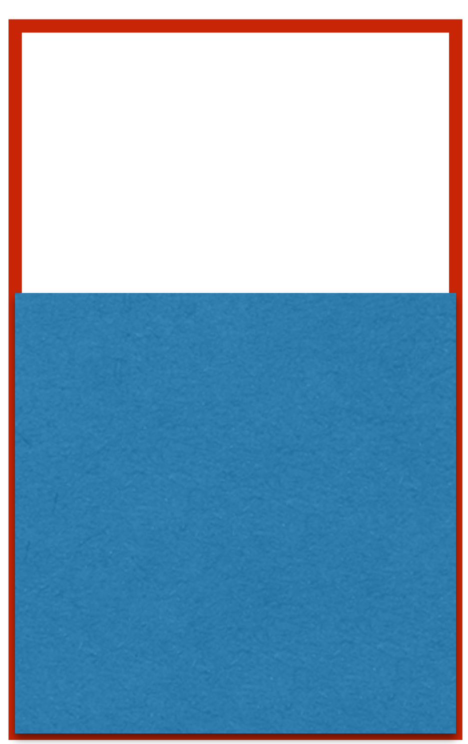
$$\Longrightarrow a^2 + b^2 = c^2$$

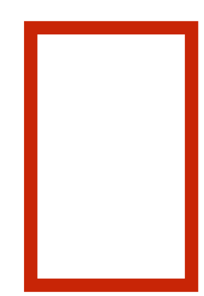
$$\Longrightarrow \square + \square = \square$$

Ancient India

- **2. Squaring a rectangle.** Meaning: Given a rectangle, create a square whose area is equal to the area of the rectangle.

1. Create Square in the bottom of the rectangle.

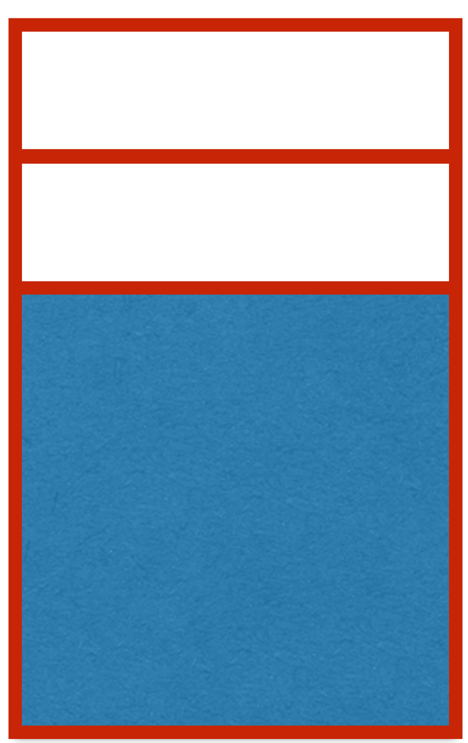


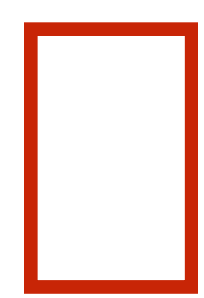
Total Area = 

Ancient India

- **2. Squaring a rectangle.** Meaning: Given a rectangle, create a square whose area is equal to the sum of the other two.

2. Cut the top region in half

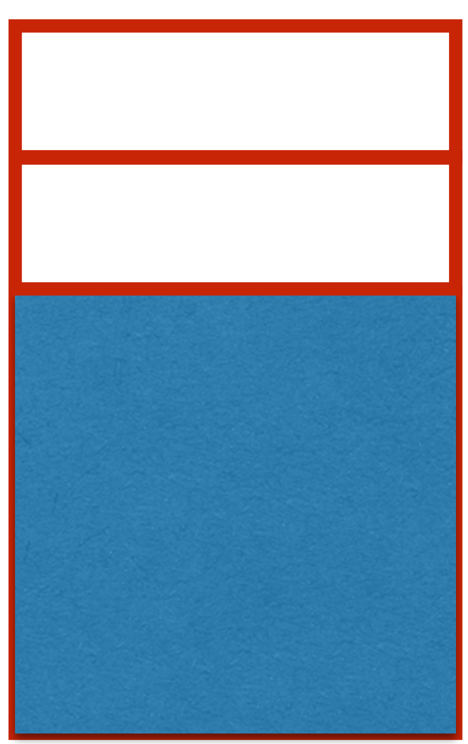


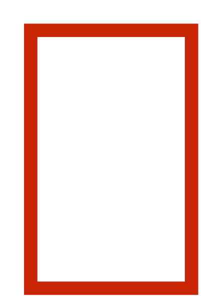
Total Area = 

Ancient India

- **2. Squaring a rectangle.** Meaning: Given a rectangle, create a square whose area is equal to the sum of the other two.

3. Move the top-most rectangle to the right side of the square

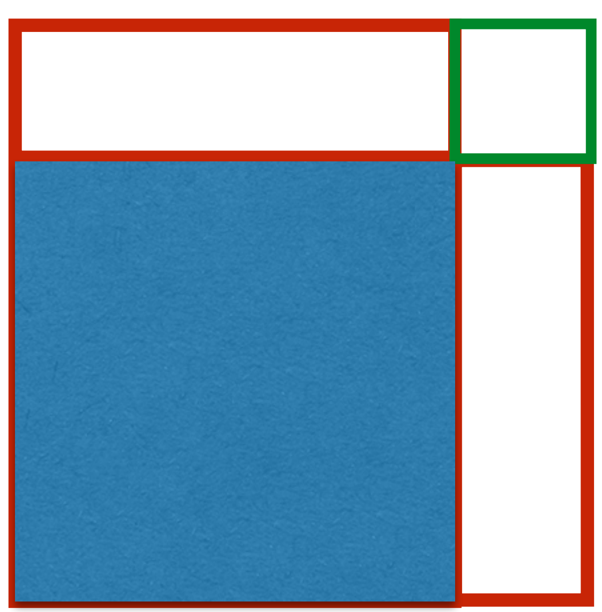


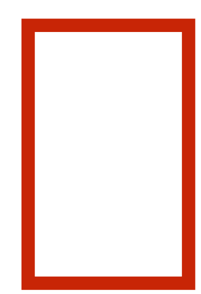
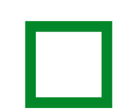
Total Area = 

Ancient India

- **2. Squaring a rectangle.** Meaning: Given a rectangle, create a square whose area is equal to the sum of the other two.

4. Fill in the gap to form a square.



Total Area =  + 

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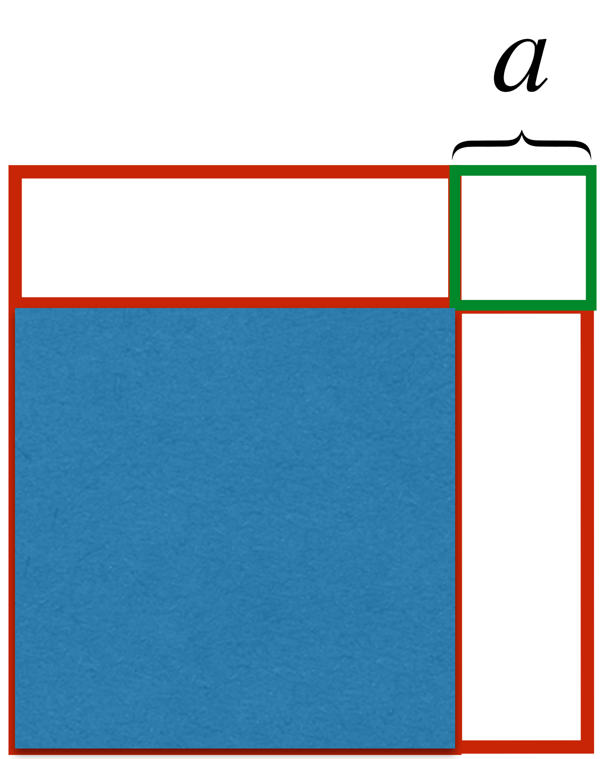
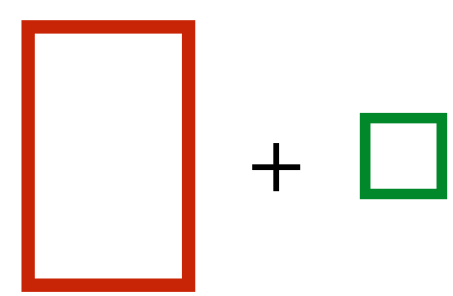


Total Area = 

Ancient India

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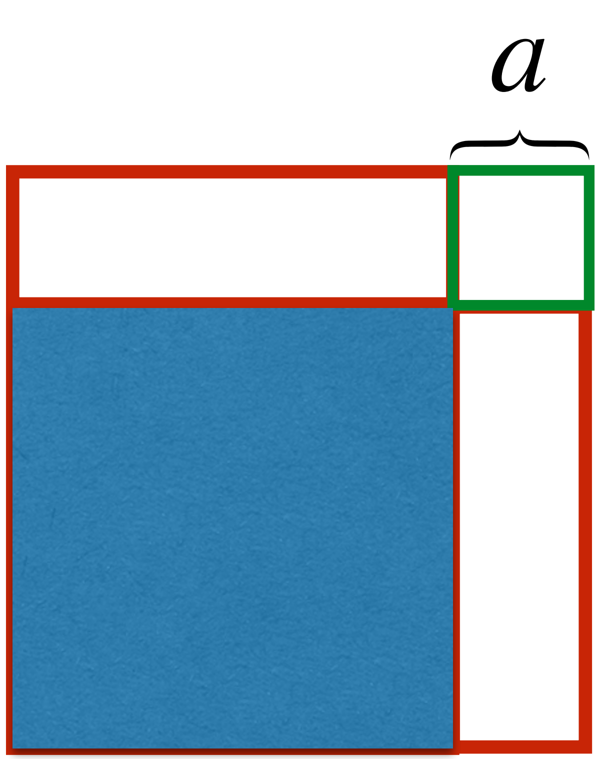
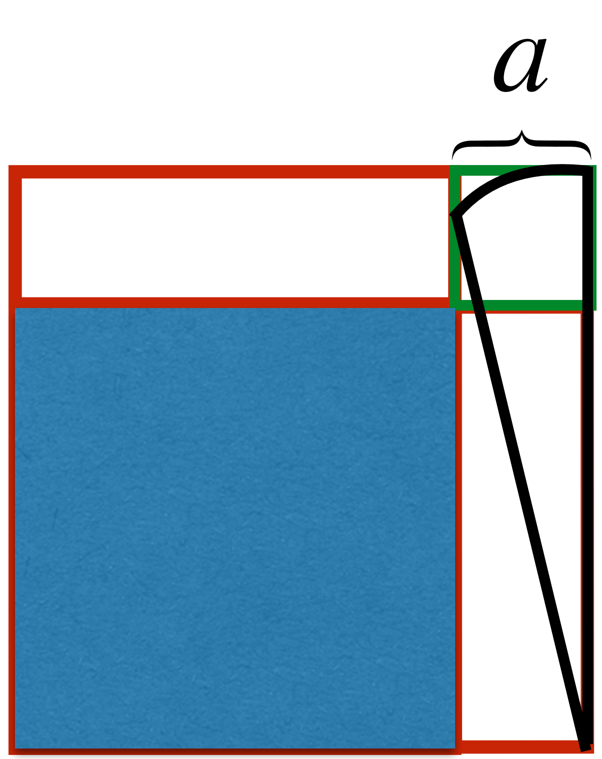


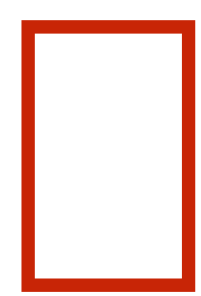
Total Area = □ + a^2

Ancient India

- **2. Squaring a rectangle.** Meaning: Given a rectangle, create a square whose area is equal to the sum of the other two.

5. Using a compass, make the arc shown.



Total Area =  + a^2

Ancient India

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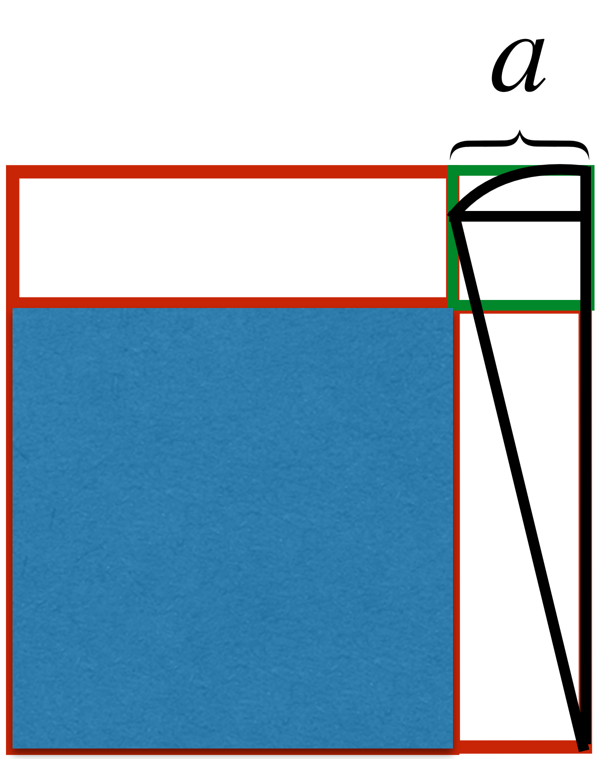
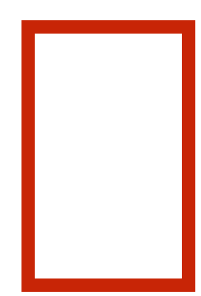


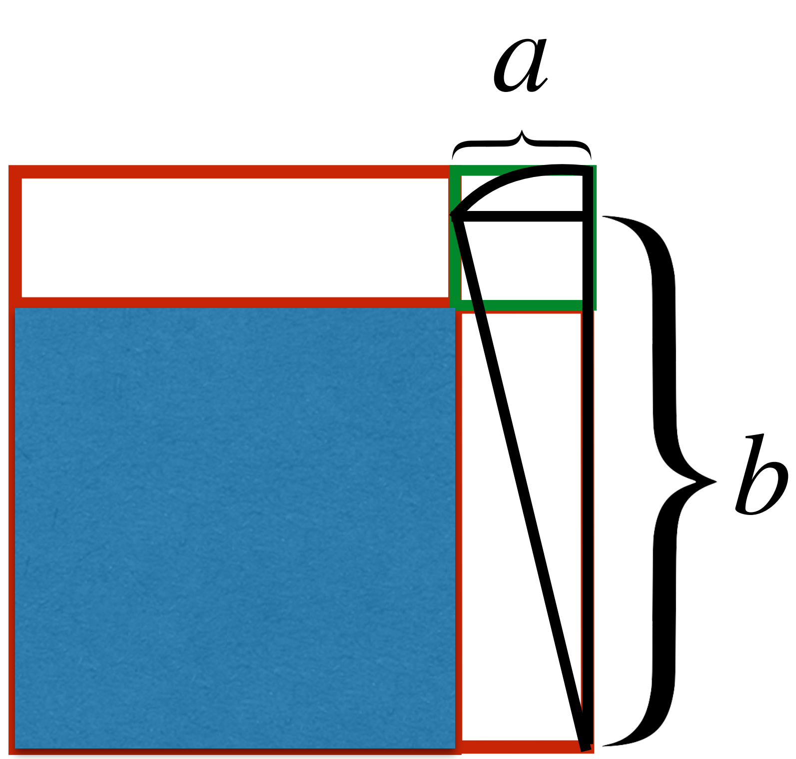
6. Draw the following horizontal line

Total Area =  + a^2

Ancient India

- **2. Squaring a rectangle.** Meaning: Given a rectangle, create a square whose area is equal to the sum of the other two.

7. Label this length b, and let the hypotenuse be c:



Total Area = $\square + a^2$

Ancient India

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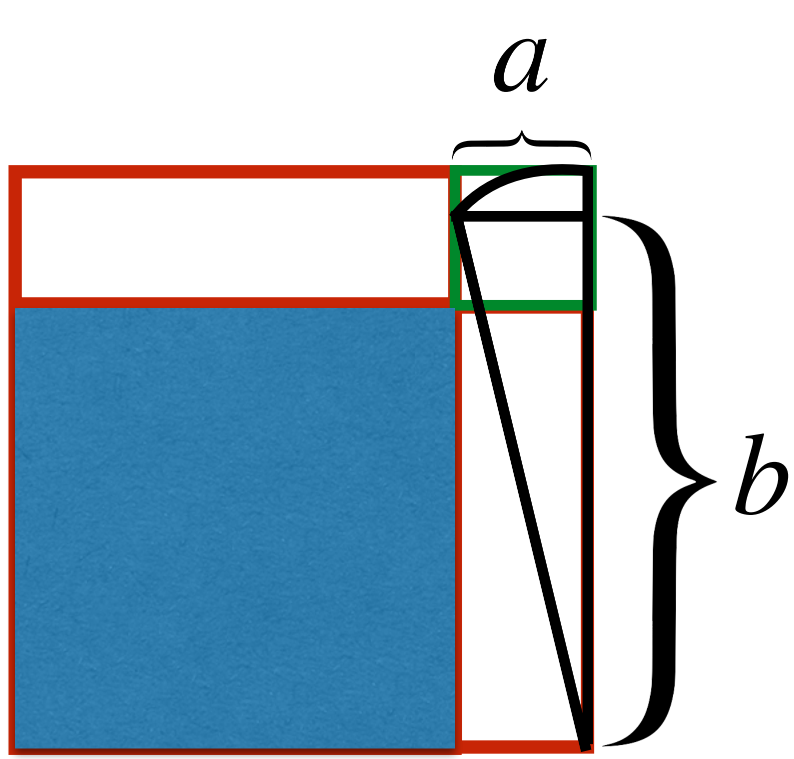

$$c^2 = \square + a^2$$

Ancient India

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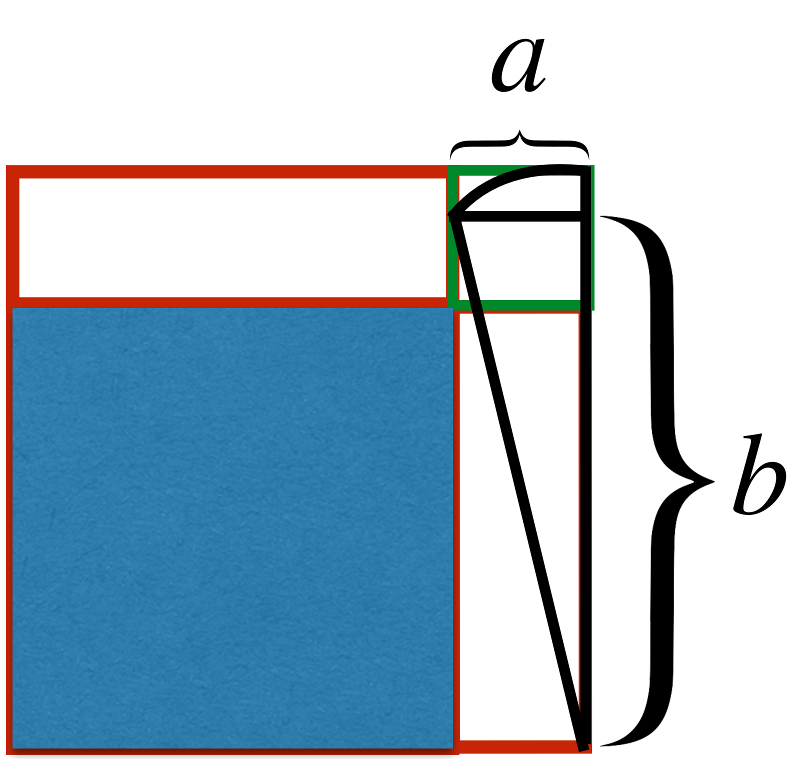
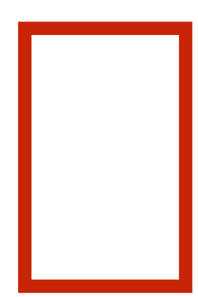
7. Label this length b, and let the hypotenuse be c:





$= c^2 - a^2$

Ancient India

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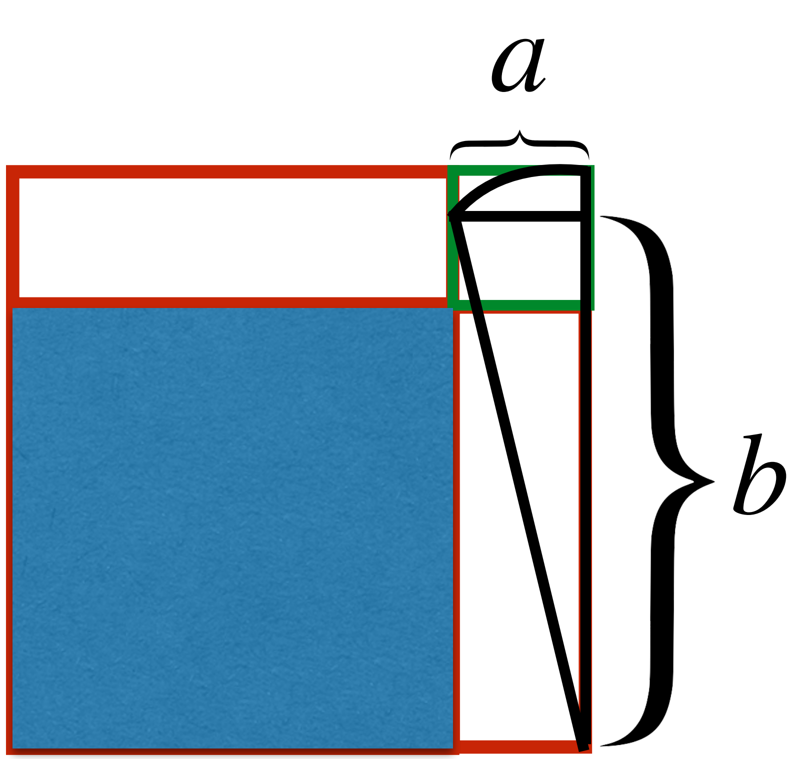
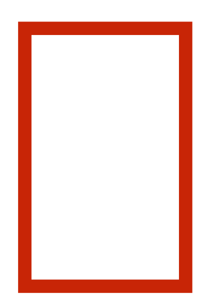


By the Pythagorean theorem,

$$a^2 + b^2 = c^2.$$

$$\implies b^2 = c^2 - a^2.$$



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Ancient India

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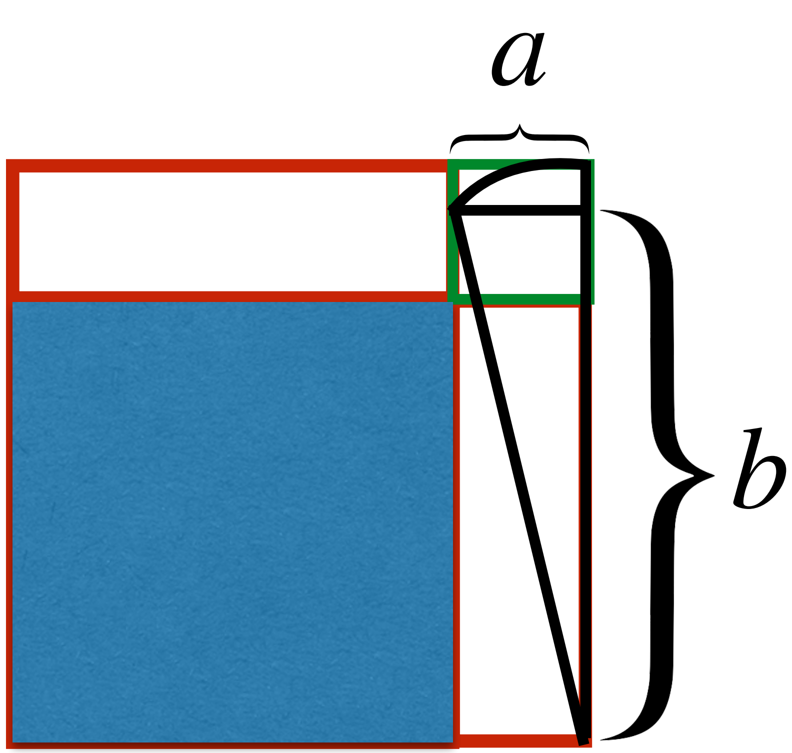
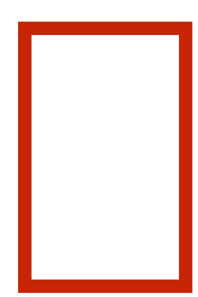


7. Label this length b, and let the hypotenuse be c:

By the Pythagorean theorem,

$$a^2 + b^2 = c^2.$$

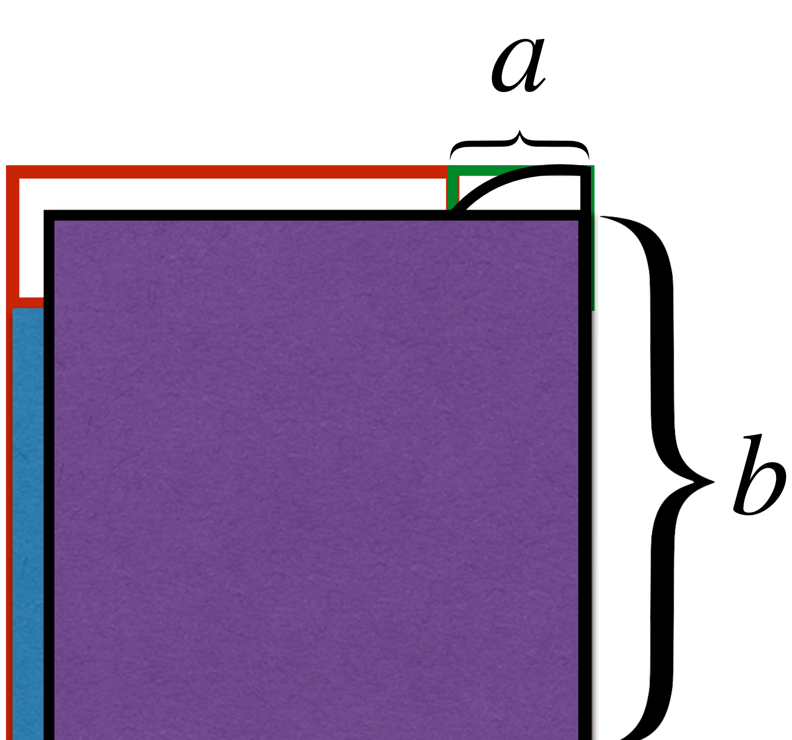
$$\implies b^2 = c^2 - a^2.$$

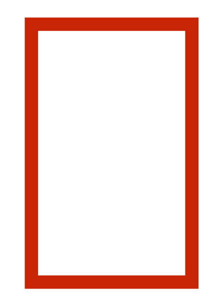
 $= b^2$

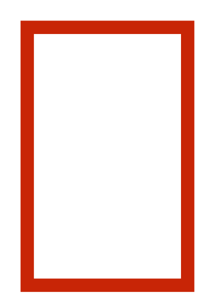
Ancient India

- **2. Squaring a rectangle.** Meaning: Given a rectangle, create a square whose area is equal to the sum of the other two.

8. Build a square on this line segment. The area of this square equals b^2.



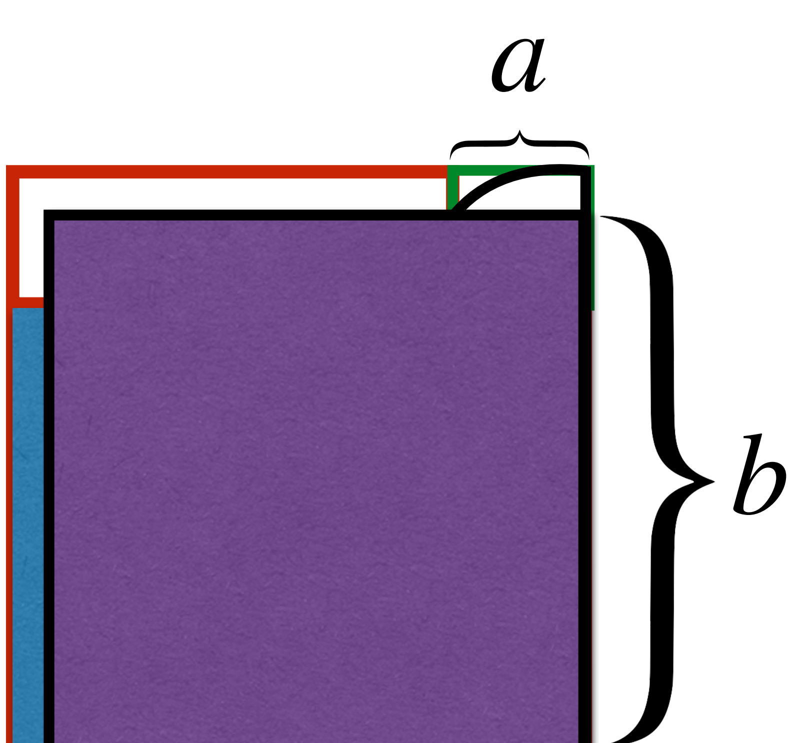
So,  = 

 $= b^2$

Ancient India

- **2. Squaring a rectangle.** Meaning: Given a rectangle, create a square whose area is equal to the sum of the other two.

9. So, this square has the same area as the original rectangle.



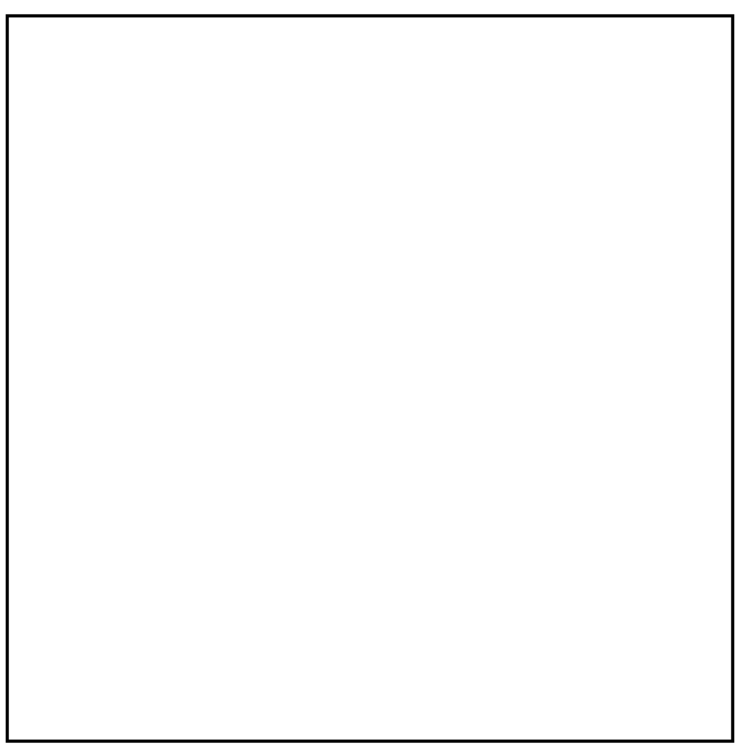
Ancient India

- **3. Circling a square or squaring a circle.** Meaning: Given a square, create a circle whose area is equal to that of the square. Or, given a circle, create a square of the same area.

- Note: While the last two gave perfect constructions, these ones do not. Also, they did not note in their work that these two were imperfect.

Ancient India

- **3. Circling a square or squaring a circle.**



Ancient India

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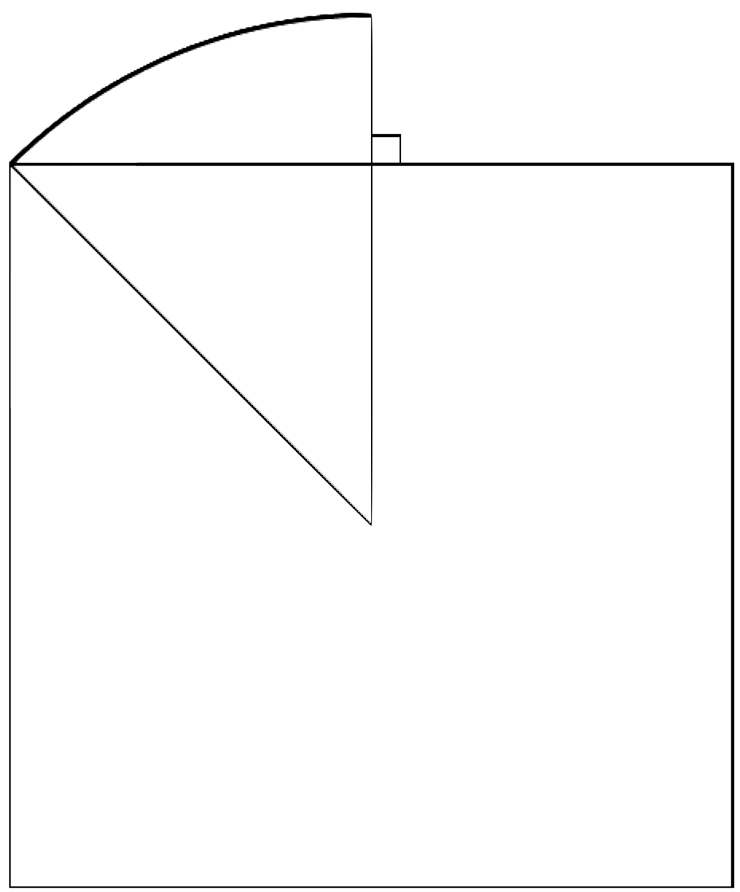

Ancient India

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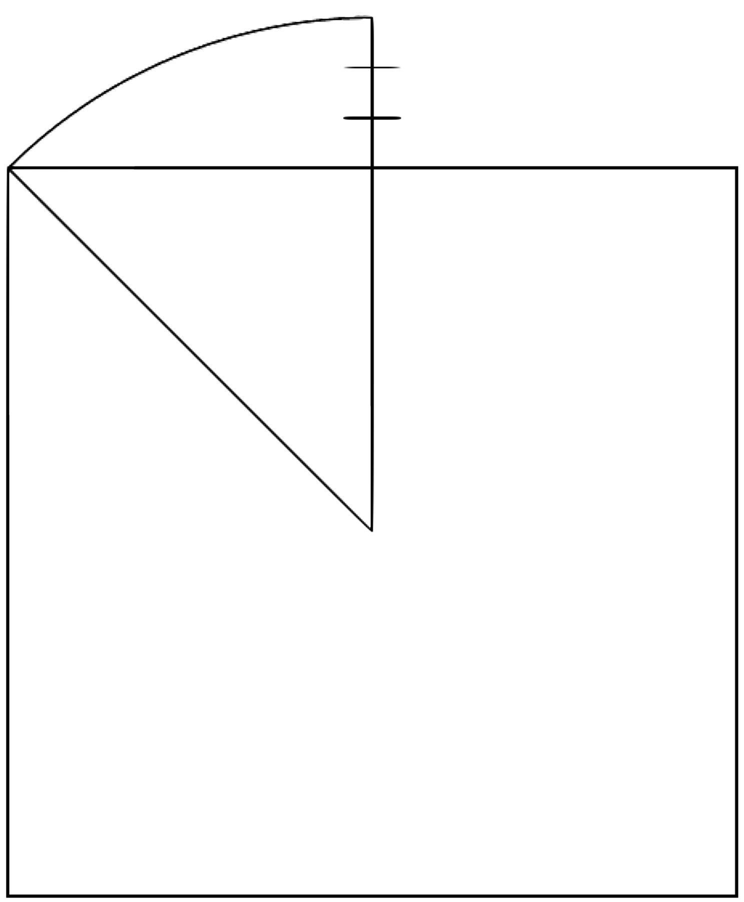

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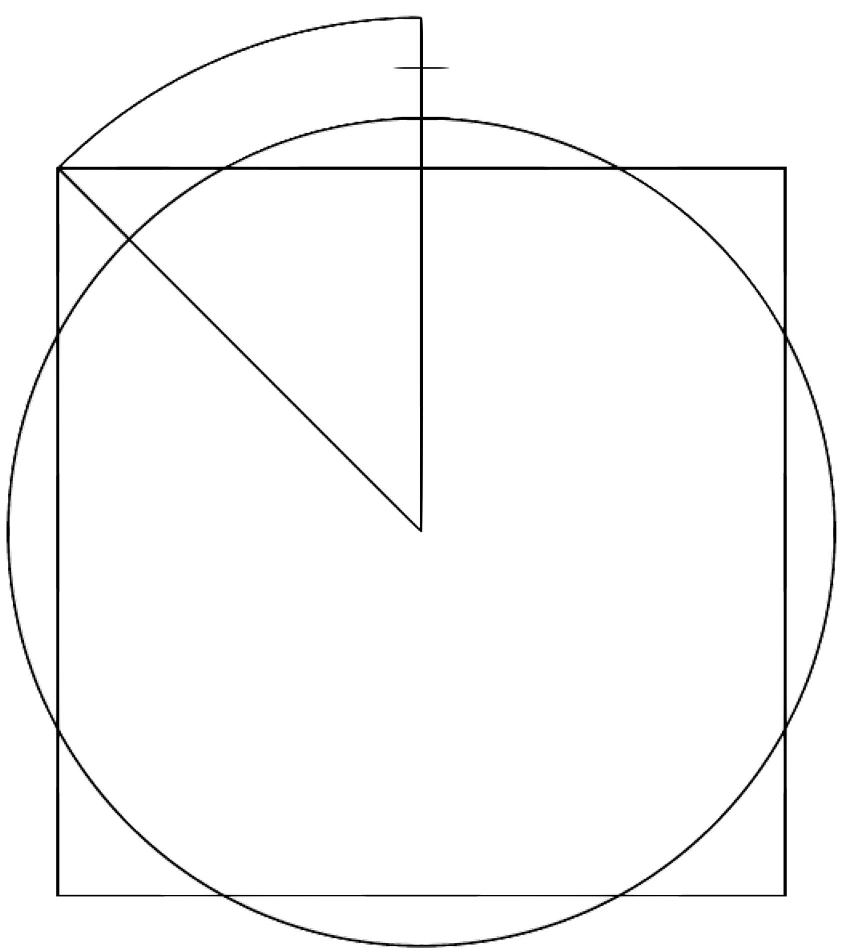

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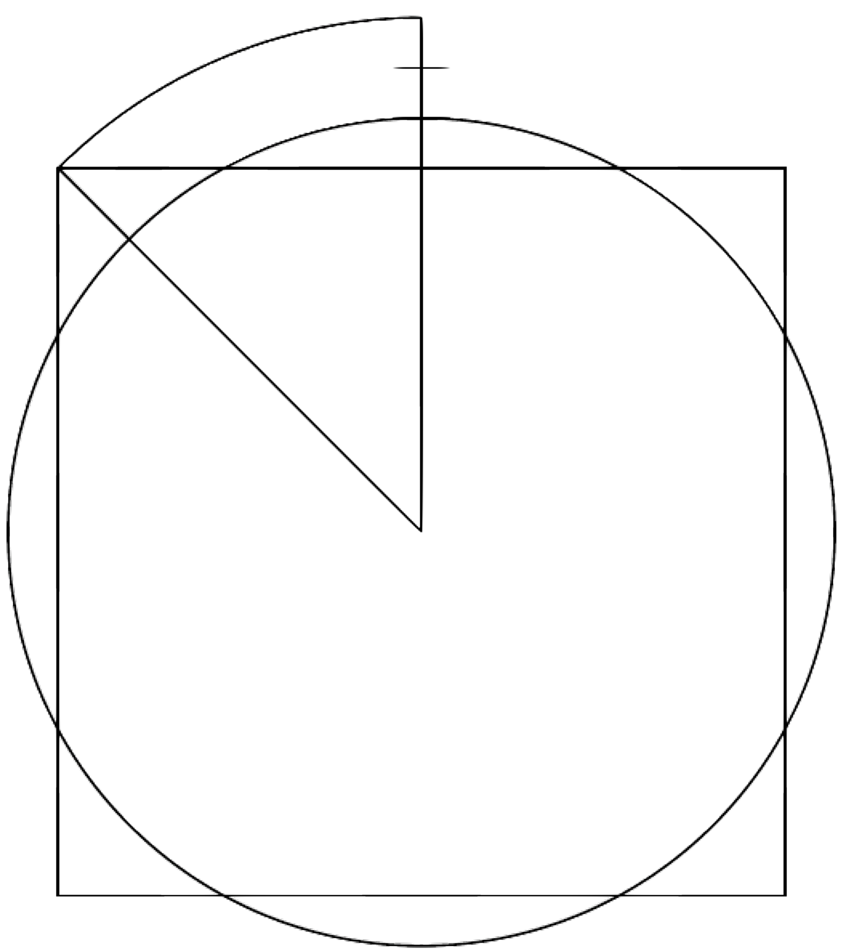

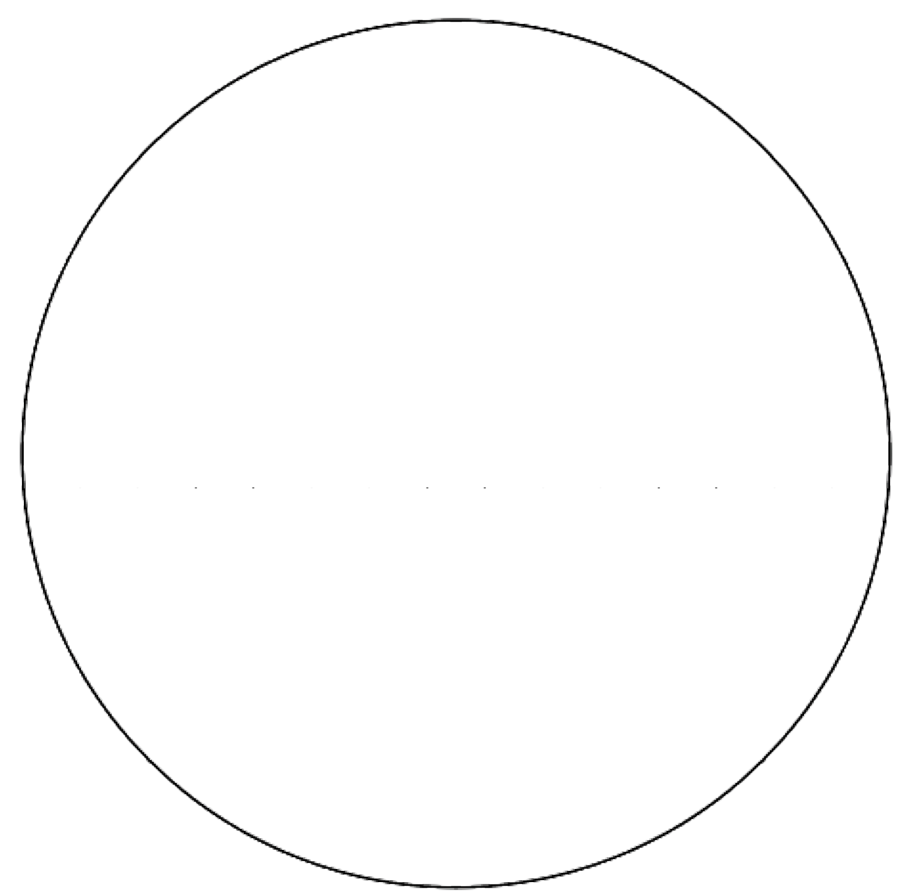

Ancient India

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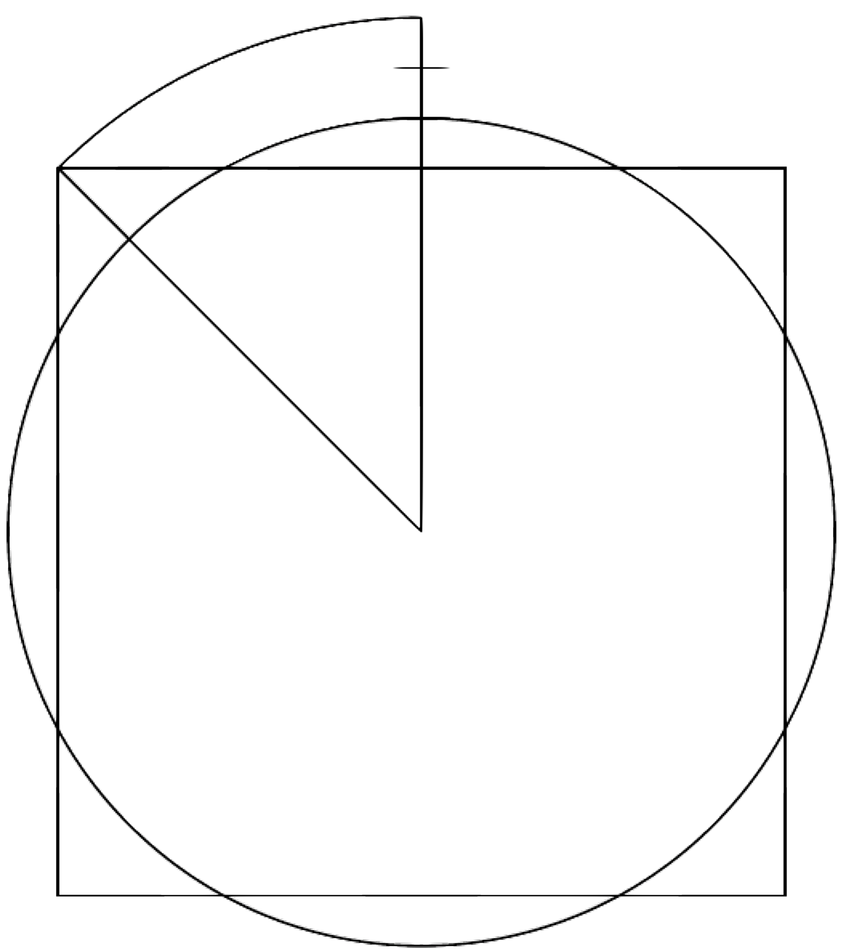

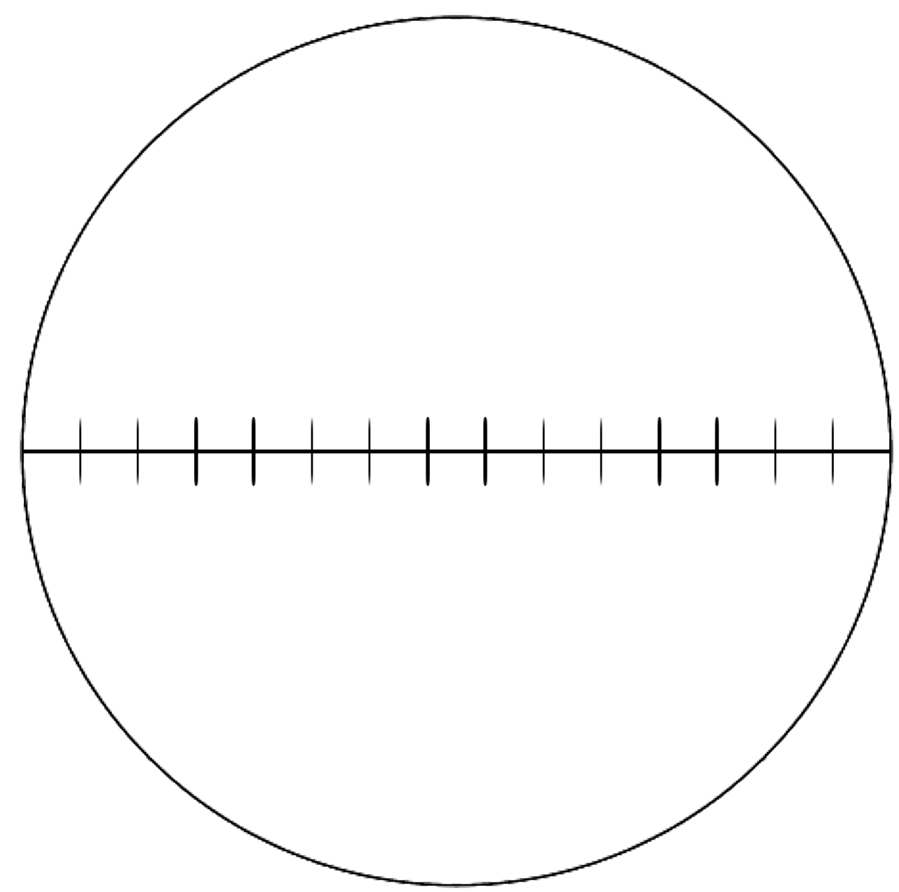

Ancient India

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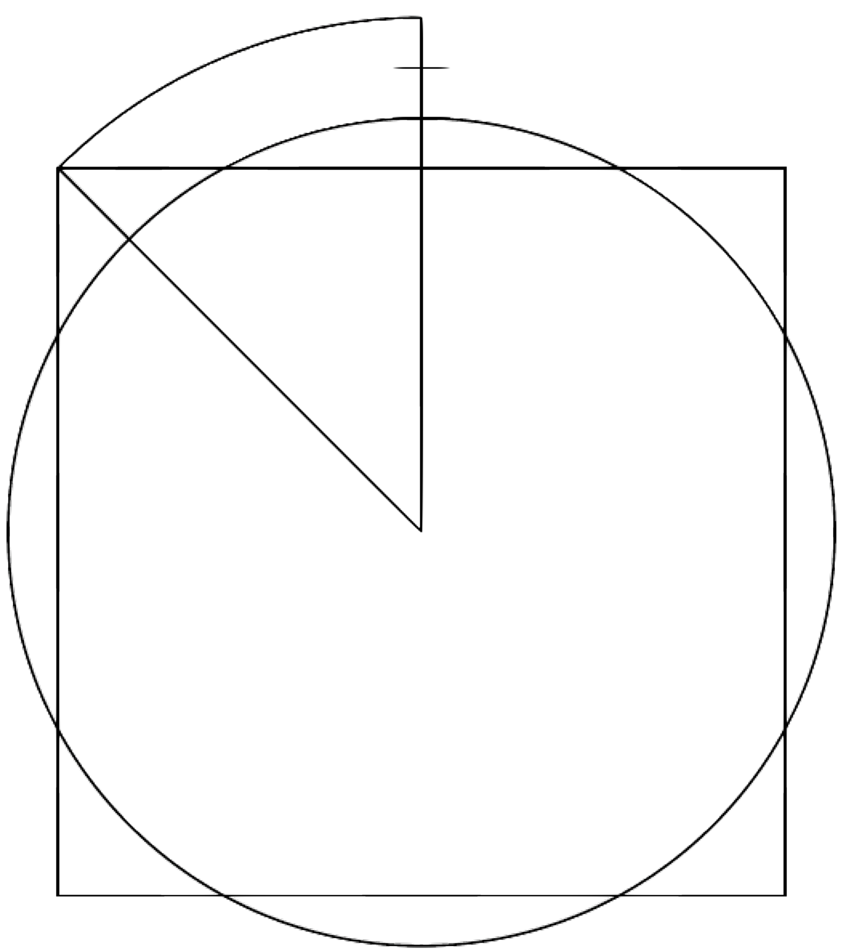
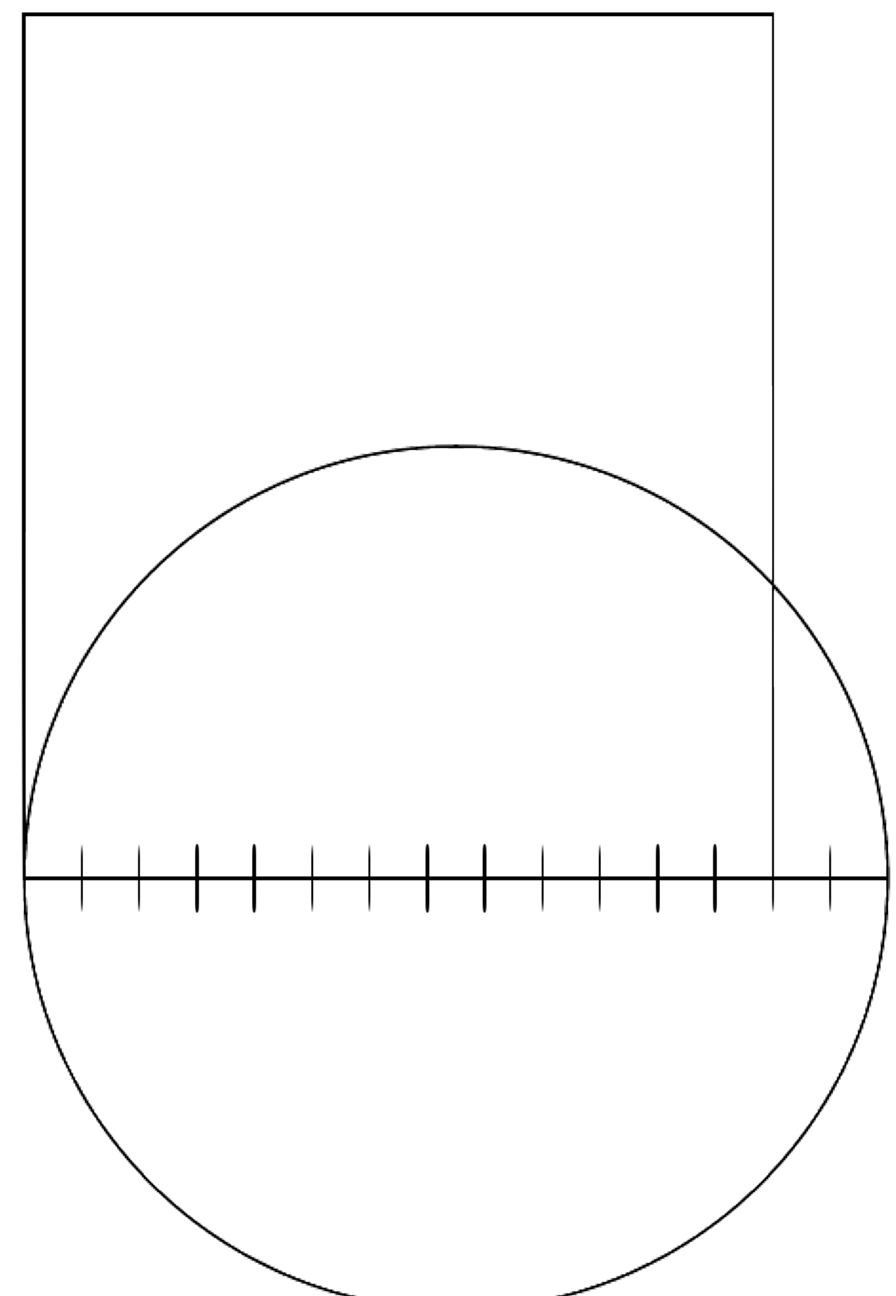


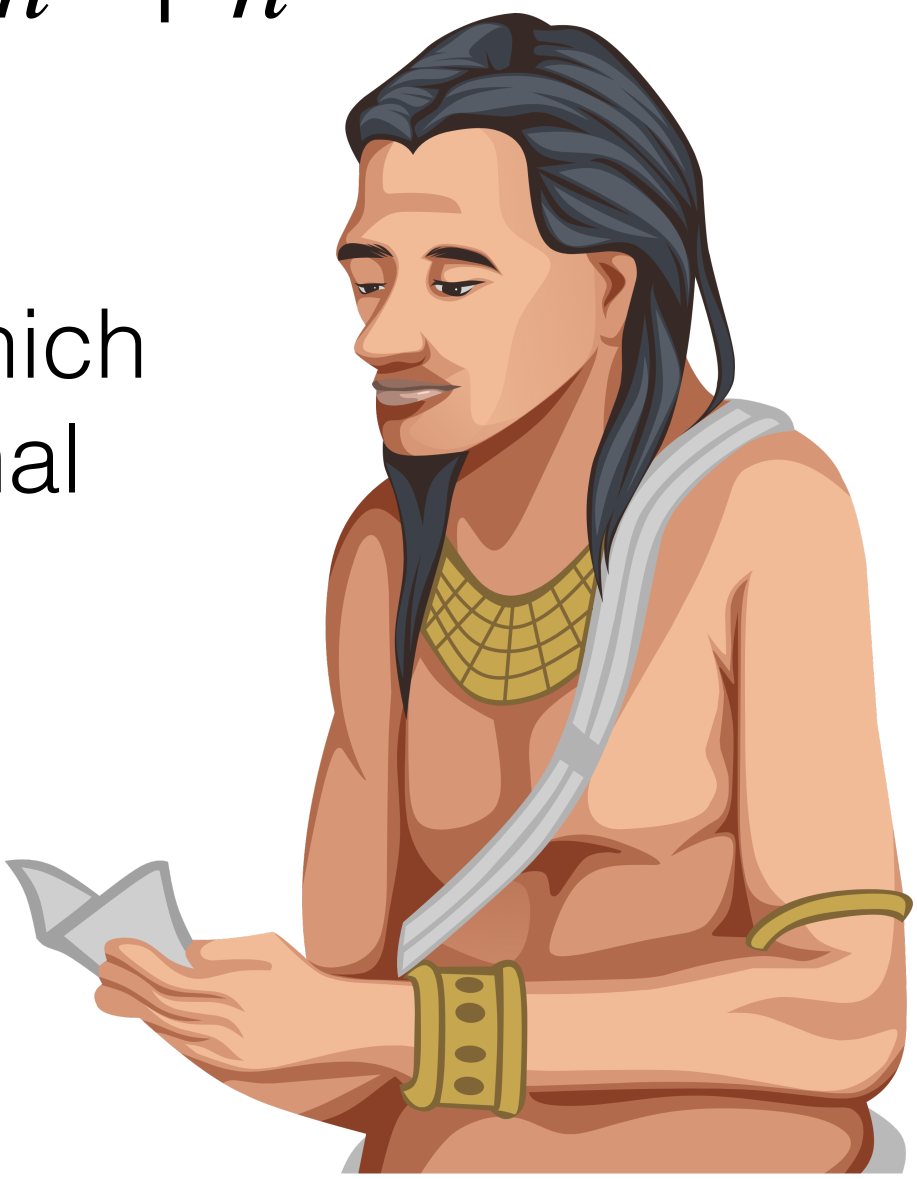


Pythagorean Triples

- Euclid characterized all *primitive Pythagorean triples*. They are all of this form:

$$a = m^2 - n^2, \quad b = 2mn, \quad c = m^2 + n^2$$

- Medieval Indians studied "rational triangles," which are triangles for which all of its sides and its area are rational numbers. Brahmagupta proved:



Theorem.

Theorem 3.3 (Brahmagupta)**.** If a triangle has rational sides a, b and c, and also has a rational area, then

$$a = \frac{u^2 + v^2}{v}, \quad b = \frac{u^2 + w^2}{w}, \quad c = \frac{u^2 - v^2}{v} + \frac{u^2 - w^2}{w},$$

for some rational numbers u, v and w.

The Aftermath

- The problem of “squaring a circle” is this: Given a circle, use a straightedge and compass to create a square of the same area.
- This was proved to be impossible in 1882. To do so, Ferdinand von Lindemann proved that π is transcendental (i.e., not the root of a polynomial with rational coefficients).
- Proof sketch given in the book. It’s a neat argument—but is subtle and better read than heard.

The Three Classical Problems of Antiquity

- Three problems which were notable drivers of ancient Greek math, as well as modern math. They were to use a straightedge and compass to:

> Eratosthenes, in his work entitled *Platonicus* relates that, when the gods proclaimed to the Delians through the oracle that, in order to get rid of a plague, they should construct an altar double that of the existing one, their craftsmen fell into great perplexity in their efforts to discover how a solid could be made the double of a similar solid; they therefore went to ask Plato about it, and he replied that the oracle meant, not that the gods wanted an altar of double the size, but that he wished, in setting them the task, to shame the Greeks for their neglect of mathematics and their contempt of geometry.

Shout-Outs!

- The Pythagoreans may have helped shape the Western liberal arts education. They focused on the *quadrivium* of arithmetic, geometry, music and astronomy. Math was centered in all these.

- Pythagoreans were also said to be pioneers in the math of music.

Shout-outs!

- A more modern figure who prominently weaved together math and mysticism is Englishman John Dee. In fact, during his life in England, math was considered by many to be pseudo-magical, and in 1555 Dee was arrested for the crime of "calculating."

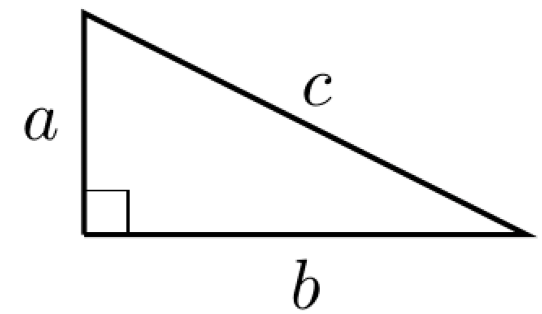
- "Vedic Mathematics" is a neat book of arithmetic tricks, but they are not actually from the Vedas.

- Lastly: My favorite proof of the Pythagorean theorem.

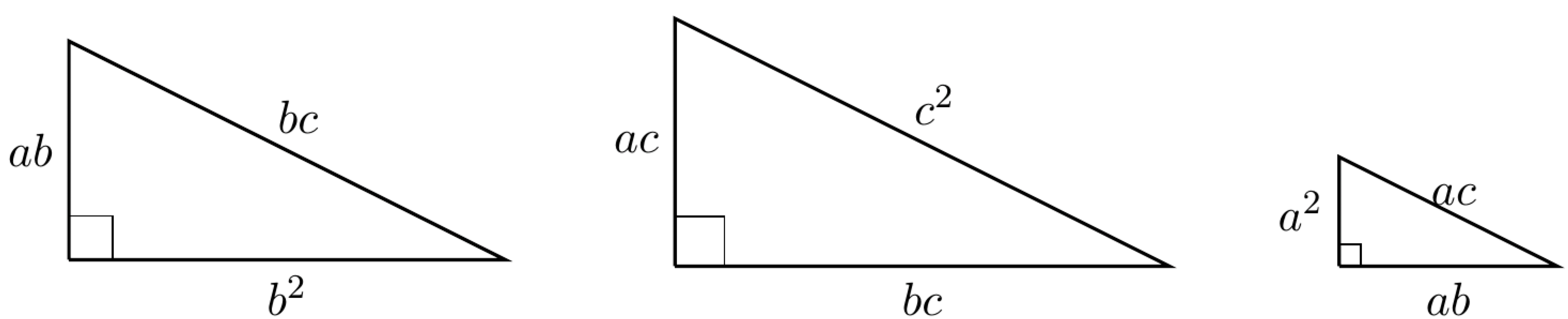
Goal: $a^2 + b^2 = c^2$

Shout-outs!

Take any right triangle:



Now, scale up this triangle three times, first by a factor of b, next by a factor of c, and last by a factor of a. This produces these three similar triangles:



<u>Goal:</u> $a^2 + b^2 = c^2$

Shout-outs!

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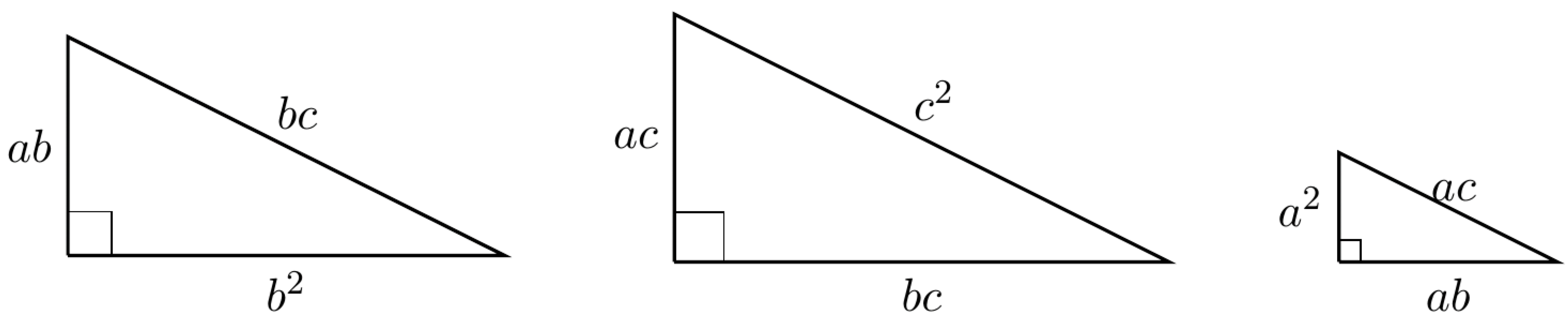
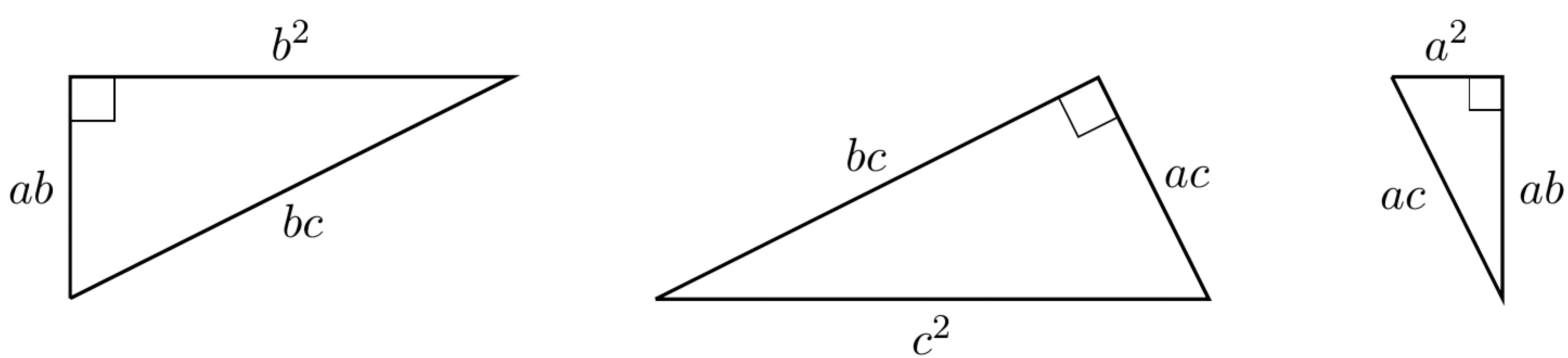


Next, by simply rotating or mirroring them, we arrive here:



Shout-outs!

Goal: $a^2 + b^2 = c^2$

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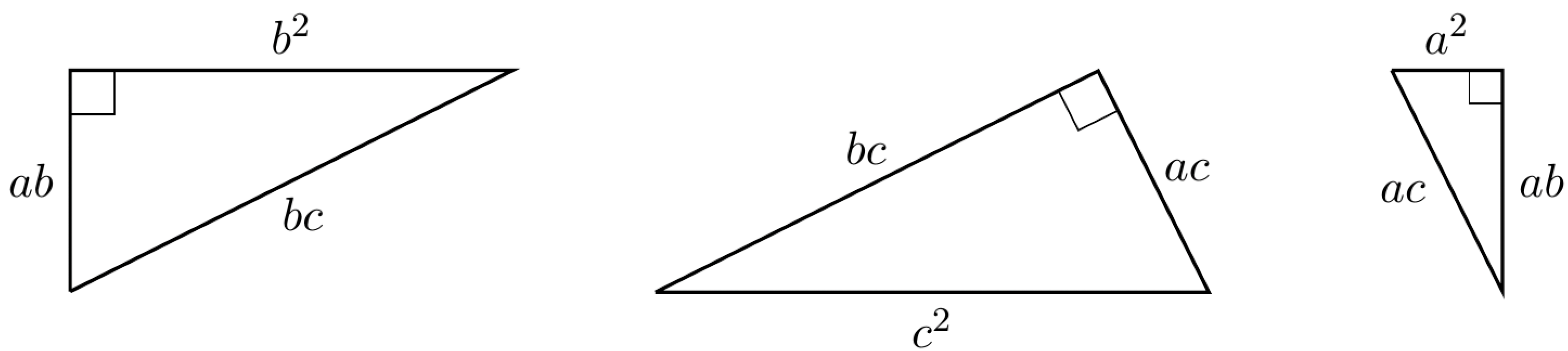
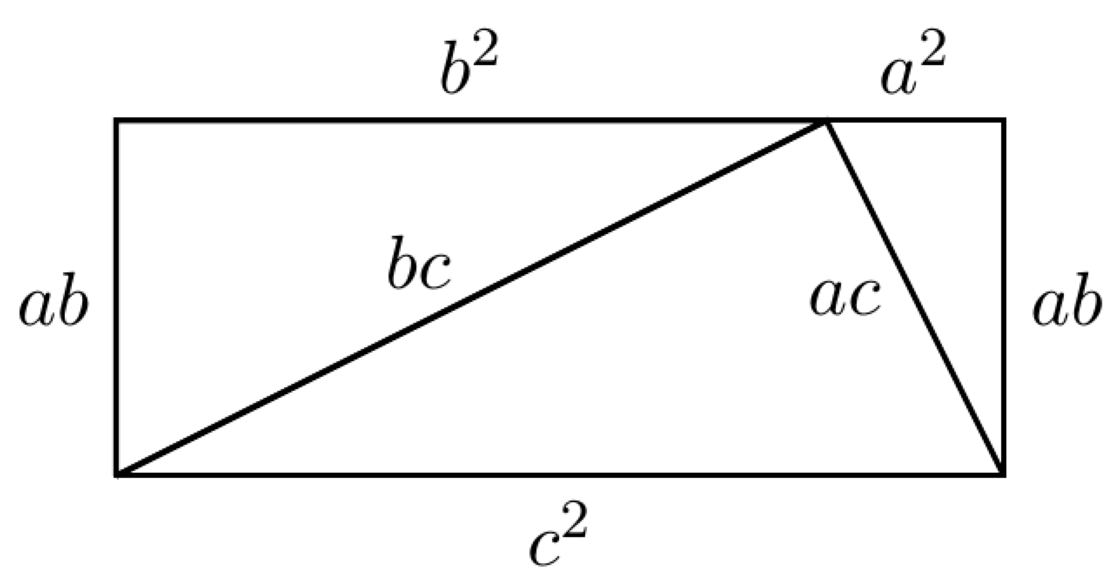


And, finally, piece them together:



Shout-outs!

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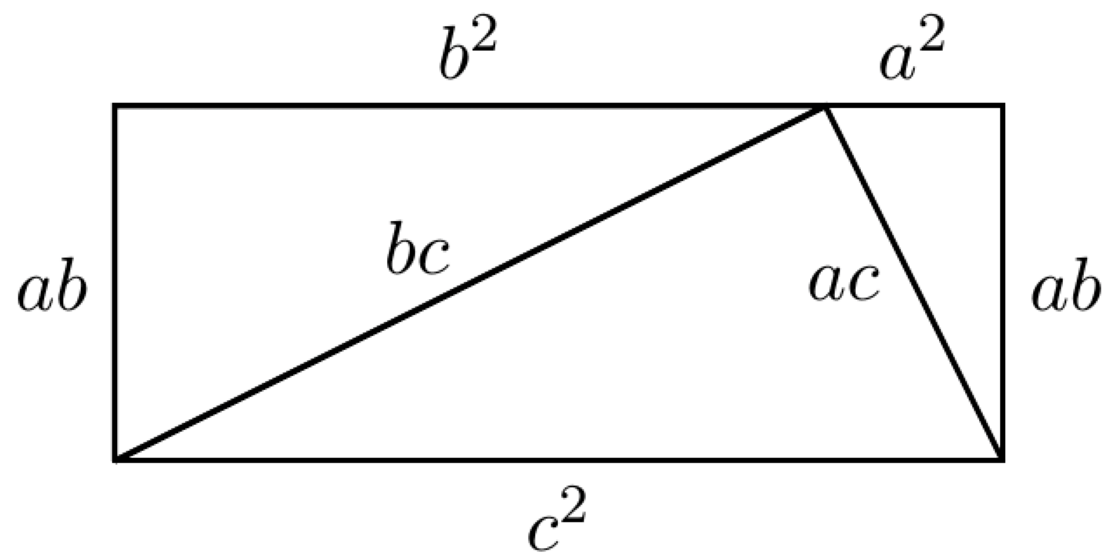

This produces a rectangle.

Utterly trivial fact: The top and bottom of a rectangle have the same length.

Thus, $a^2 + b^2 = c^2$. Q.E.D.

Goal: $a^2 + b^2 = c^2$

Shout-outs!

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Thus, $a^2 + b^2 = c^2$. Q.E.D.

People’s History

People's History of Surveying

- Math was not always seen as a field for academics. E.g., John Wallis, who was appointed the Savilian Chair of geometry at Oxford in 1649 wrote:

> My brother …taught me what he had been learning in those 3 months; that is, the *Practical* part of *Common Arithmetick* …This was my first entry into *Mathematicks*, and all the *Teaching* I had. …I did thenceforth prosecute it, (at School and in the University) not as a formal Study, but as a pleasing Diversion, at spare hours; as books of *Arithmetick*, or others *Mathematical* fel occasionally in my way. …For Mathematicks, (at that time, with us) was scarce looked upon as *Academical* Studies, but rather *Mechanical*; as the business of *Traders*, *Merchants*, *Seamen*, *Carpenters*, *Suveyors of Lands*, or the like; and perhaps *Almanack-makers in London*. And amongst more than Two hundred Students (at that time) in our College, I do not know of any Two (perhaps not any) who had more of *Mathematicks* than I, (if so much) which was then but little; And but very few, in that whole University. For the Study of *Mathematicks* was at that time more cultivated in London than in the Universities.

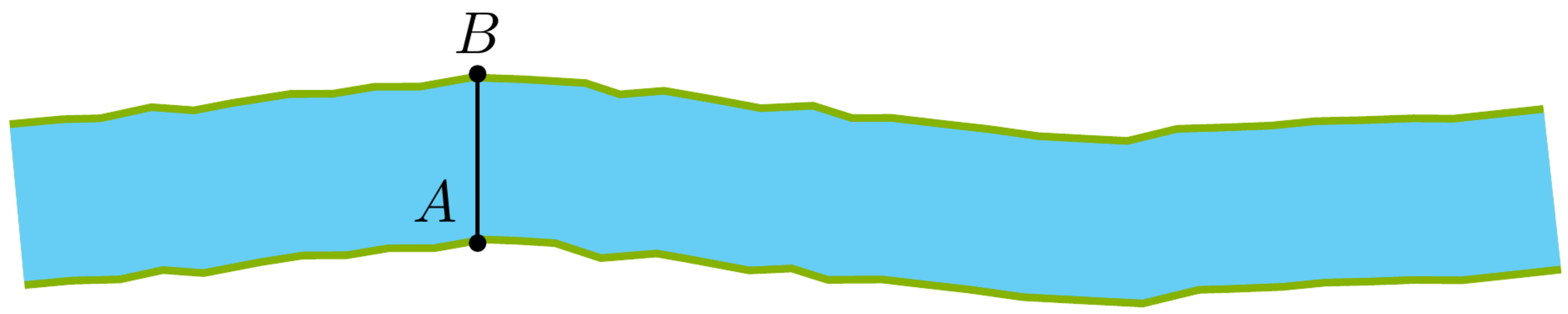
People's History of

> They said that this king (Sesostris) divided the land among all Egyptians so as to give each one a quadrangle of equal size and to draw from each his revenues, by imposing a tax to be levied yearly. But every one from whose part the river tore away anything, had to go to him and notify what had happened; he then sent the overseers, who had to measure out by how much the land had become smaller, in order that the owner might pay on what was left; in proportion to the entire tax imposed. In this way, it appears to me, geometry originated, which passed thence to Greece.

- Quote from Herodotus (430 BC):
- They probably used some interesting techniques, but those are lost. Instead, an example from Roman land surveyor Marcus Junius Nipsius.

People's History of Surveying

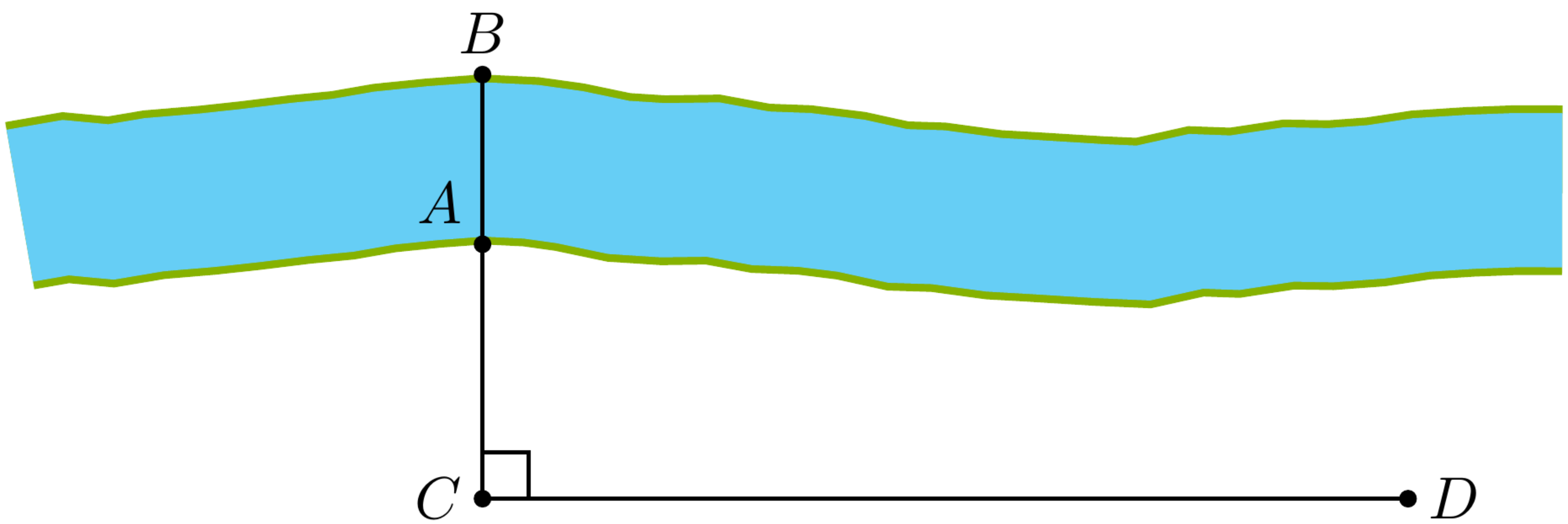
- How do you measure the distance across a river (in general, to some inaccessible point)?



- Maybe, $\boldsymbol{B}$ is some tree, or something else in sight.
- Rules: You must stay on your side of the river.

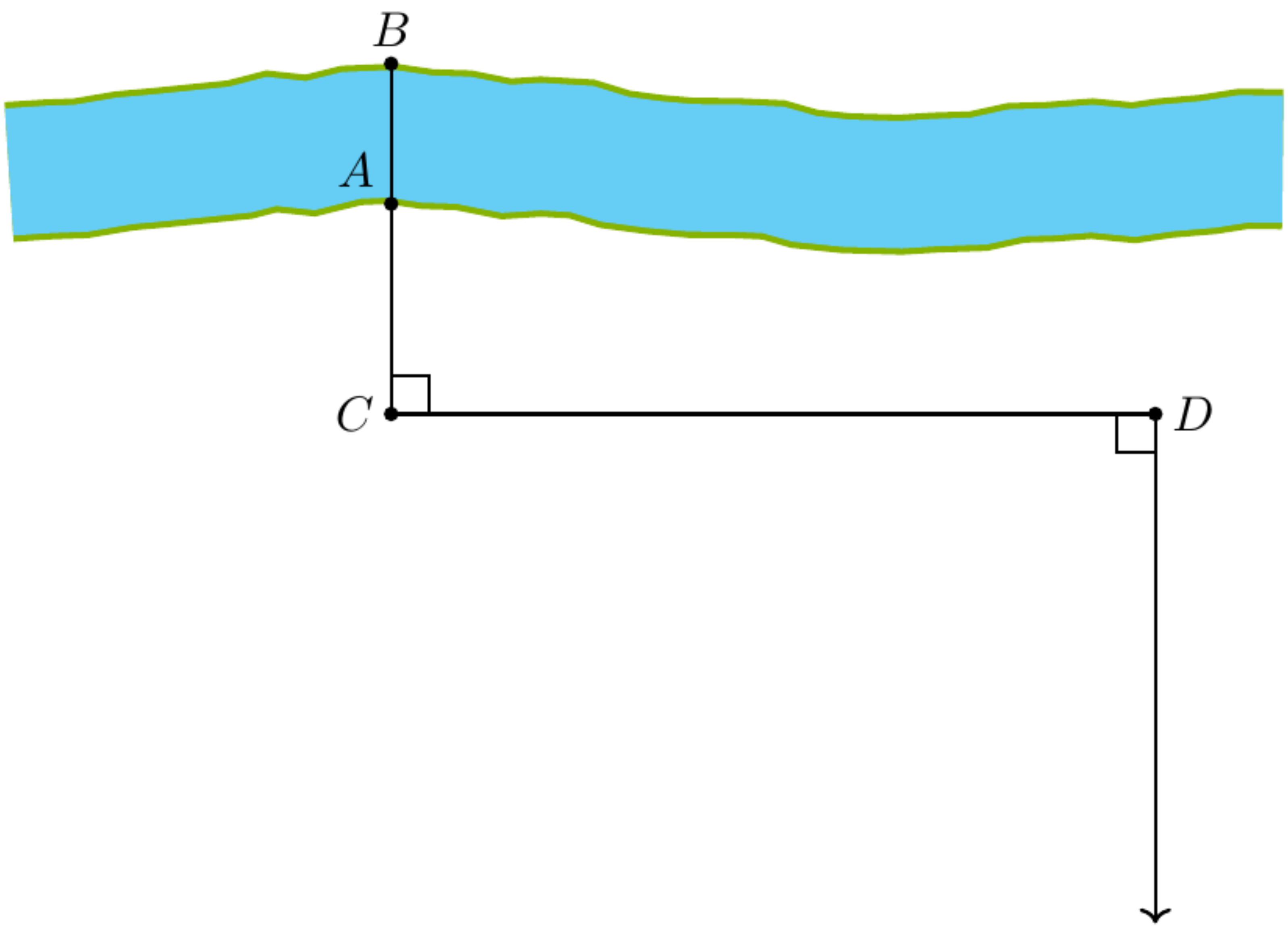
People's History of Surveying

- Idea: Extended $\overline{AB}$ to a point C. And choose a point D so that $\angle BCD$ is a right angle.



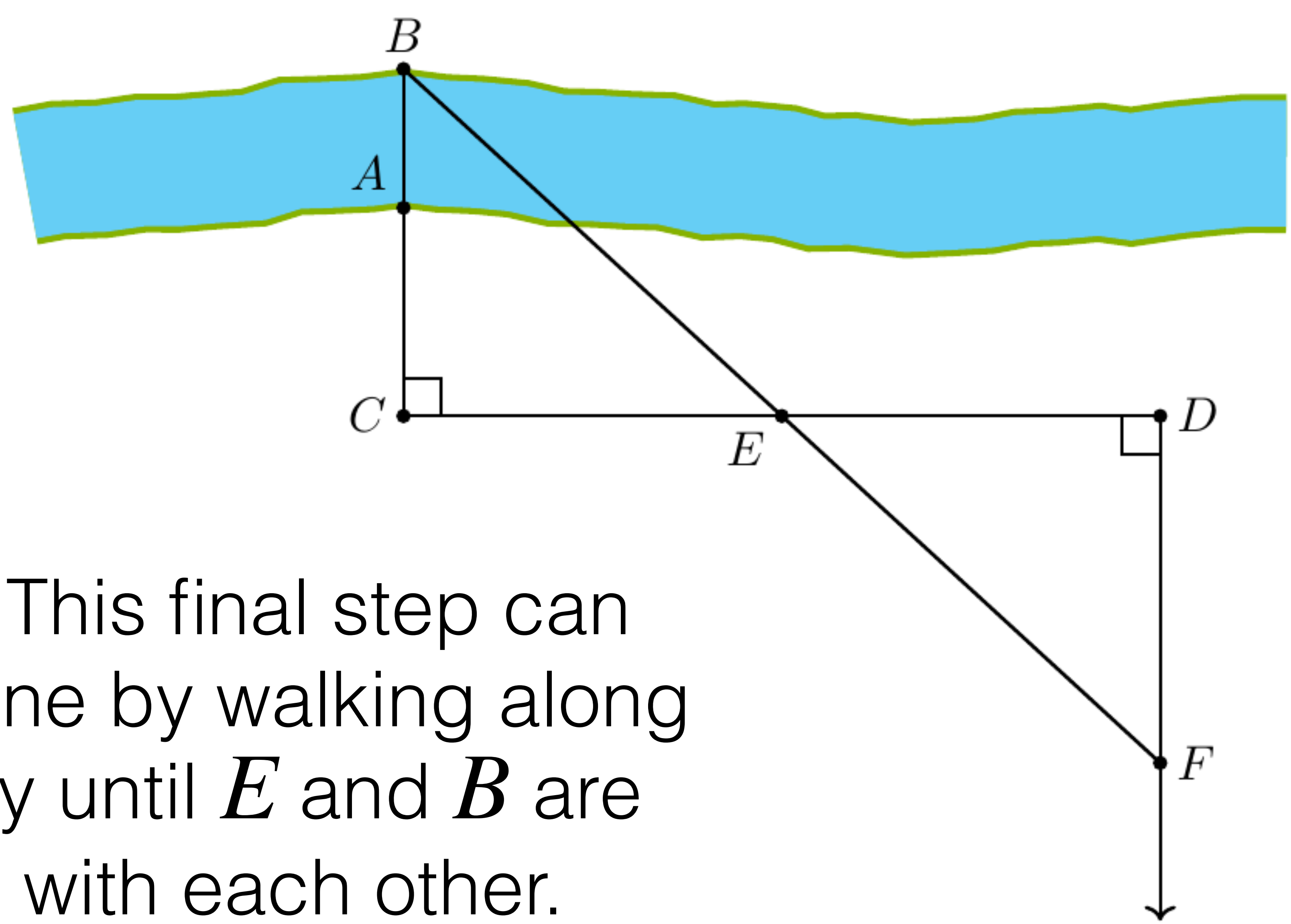
People's History of Surveying

- Make another right angle, this time at D, drawing a ray downward



People's History of Surveying

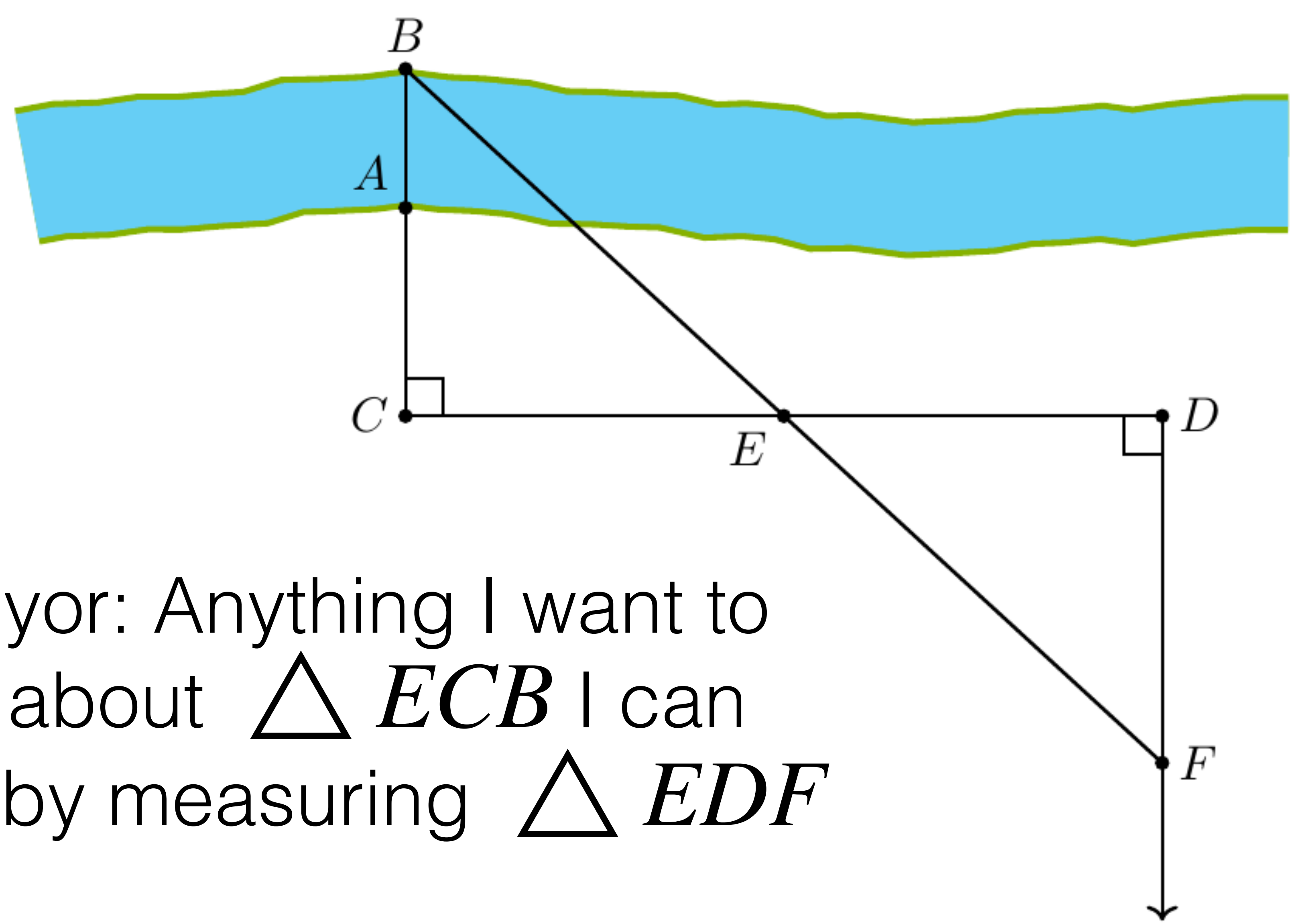
- Bisect $\overline{CD}$, call the midpoint E. Then, draw $\overline{BE}$, and continue this line until it meets the ray.



- Note: This final step can be done by walking along the ray until E and B are in line with each other.

People's History of Surveying

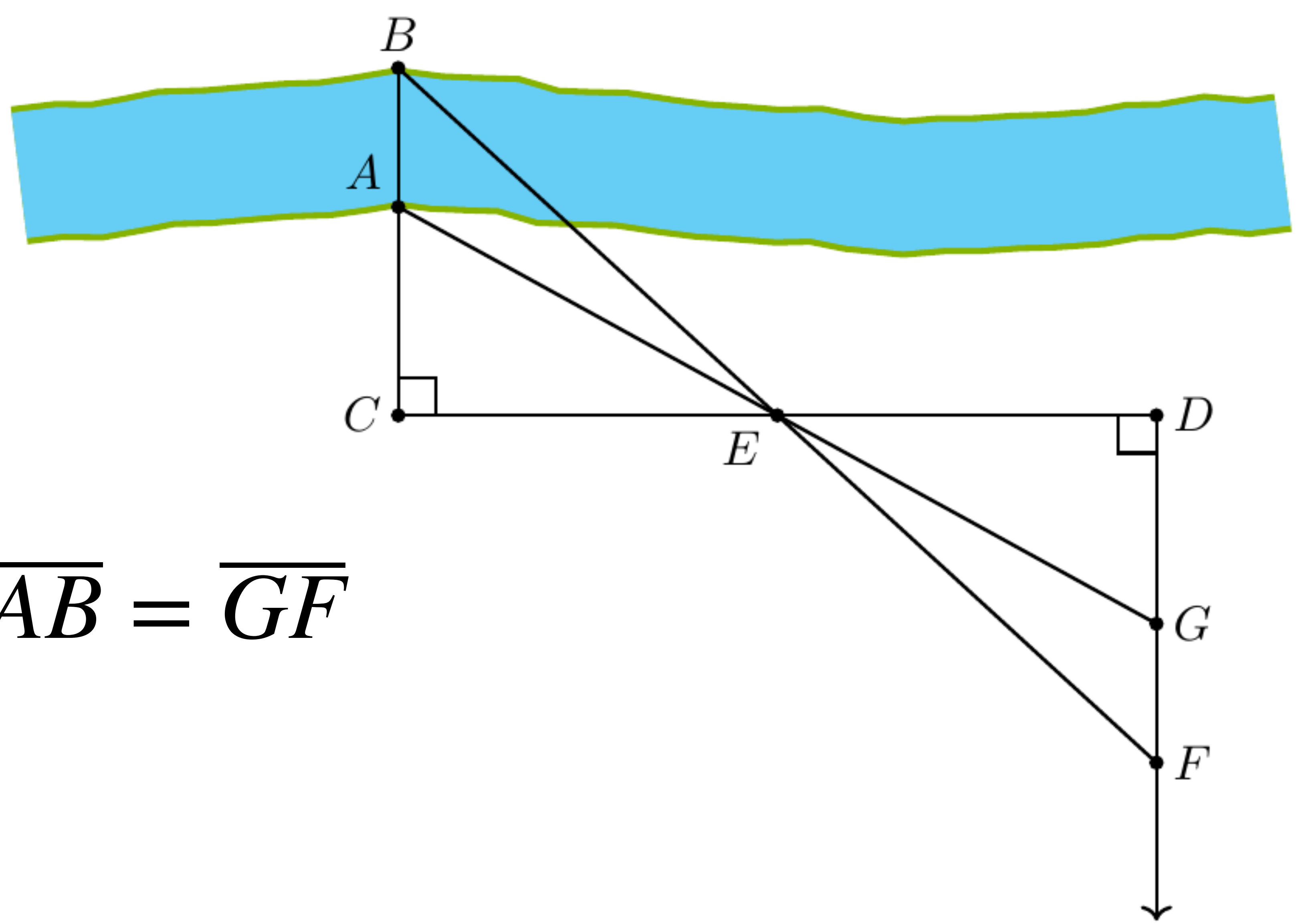
- Mathematician: $\triangle ECB$ and $\triangle EDF$ are congruent.



- Surveyor: Anything I want to know about $\triangle ECB$ I can learn by measuring $\triangle EDF$

People's History of Surveying

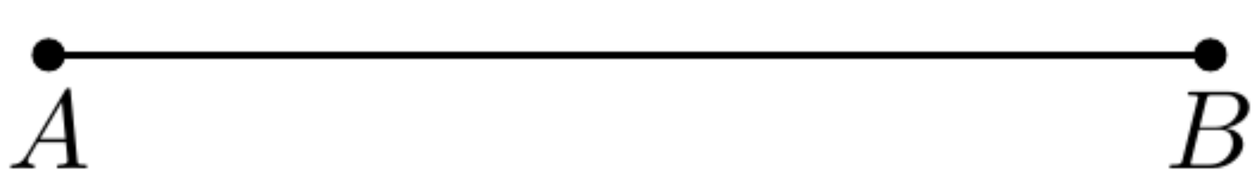
- To find the distance $\overline{AB}$, add the point G.



- Now, $\overline{AB} = \overline{GF}$
- Cool!

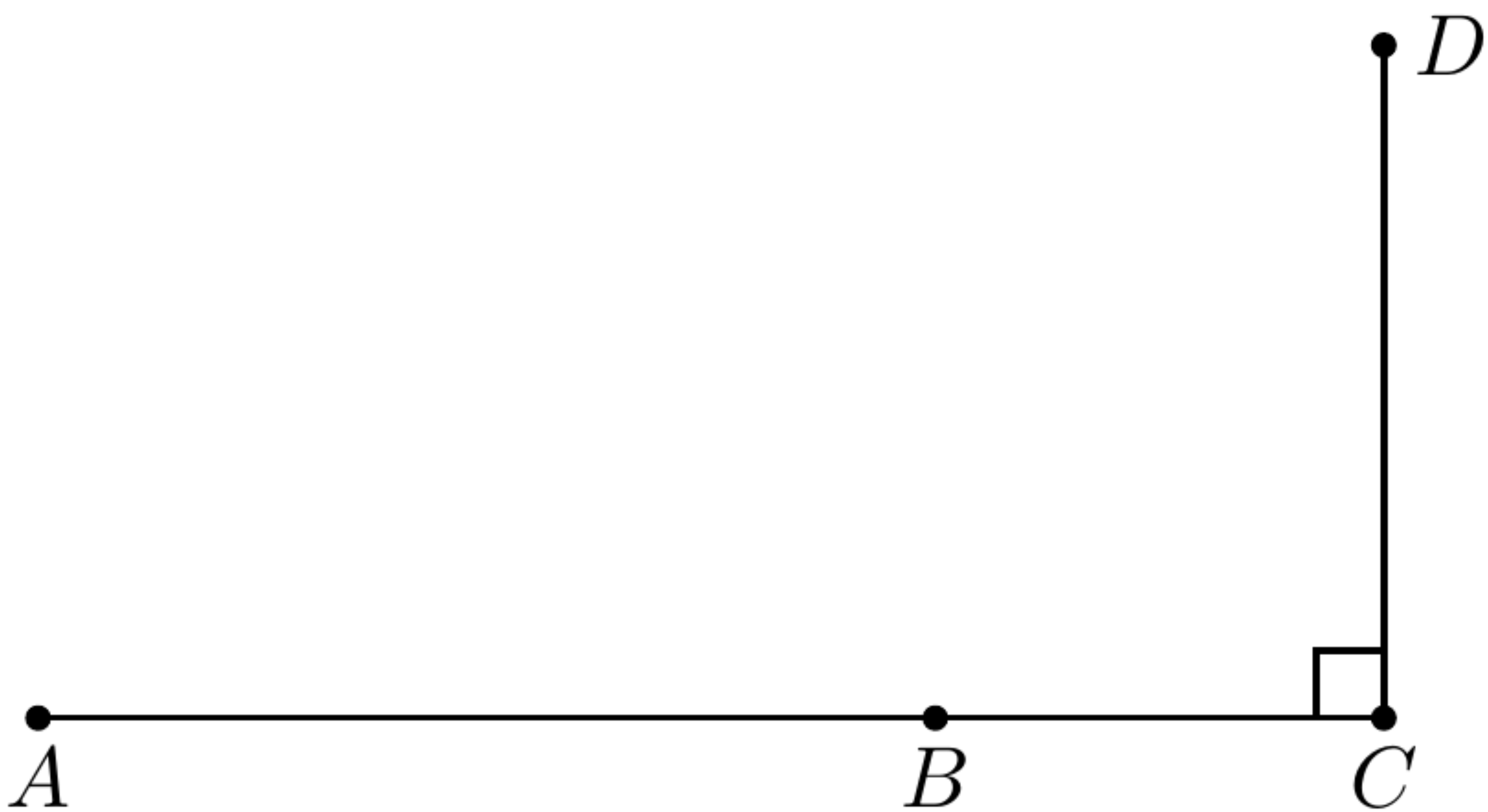
People's History of Surveying

- A related problem: An object is at an inaccessible point A. You are at B. How do you find this distance?



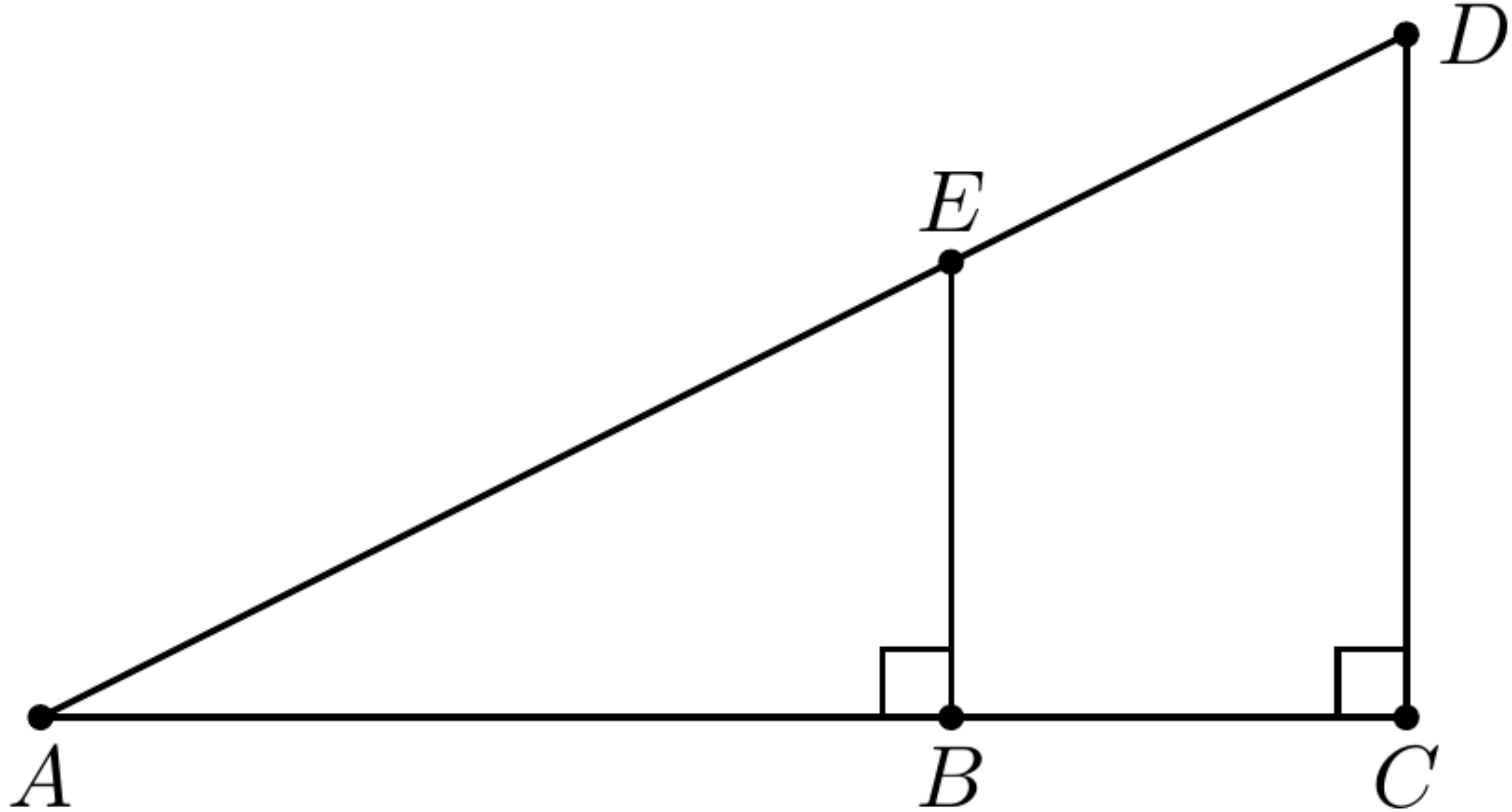
People's History of Surveying

- Extend $\overline{AB}$ to a point C, draw a perpendicular at C to a point D.



People's History of Surveying

- Walk along the perpendicular to B until you are in line with A and D.



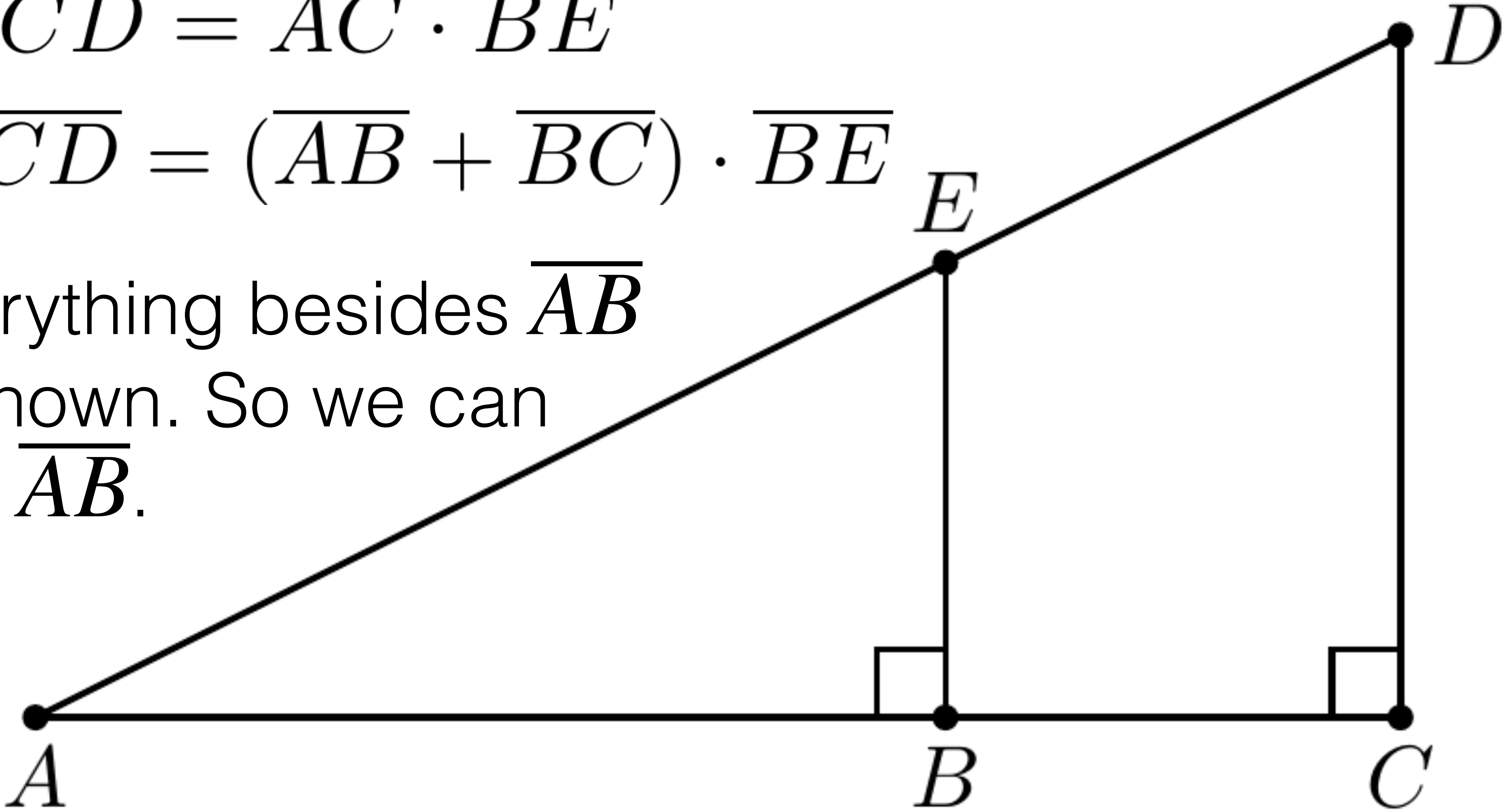
People's History of Surveying

- By similar triangles, $$\frac{\overline{AB}}{\overline{BE}} = \frac{\overline{AC}}{\overline{CD}}$$
- Thus,

$$\overline{AB} \cdot \overline{CD} = \overline{AC} \cdot \overline{BE}$$

$$\overline{AB} \cdot \overline{CD} = (\overline{AB} + \overline{BC}) \cdot \overline{BE}$$

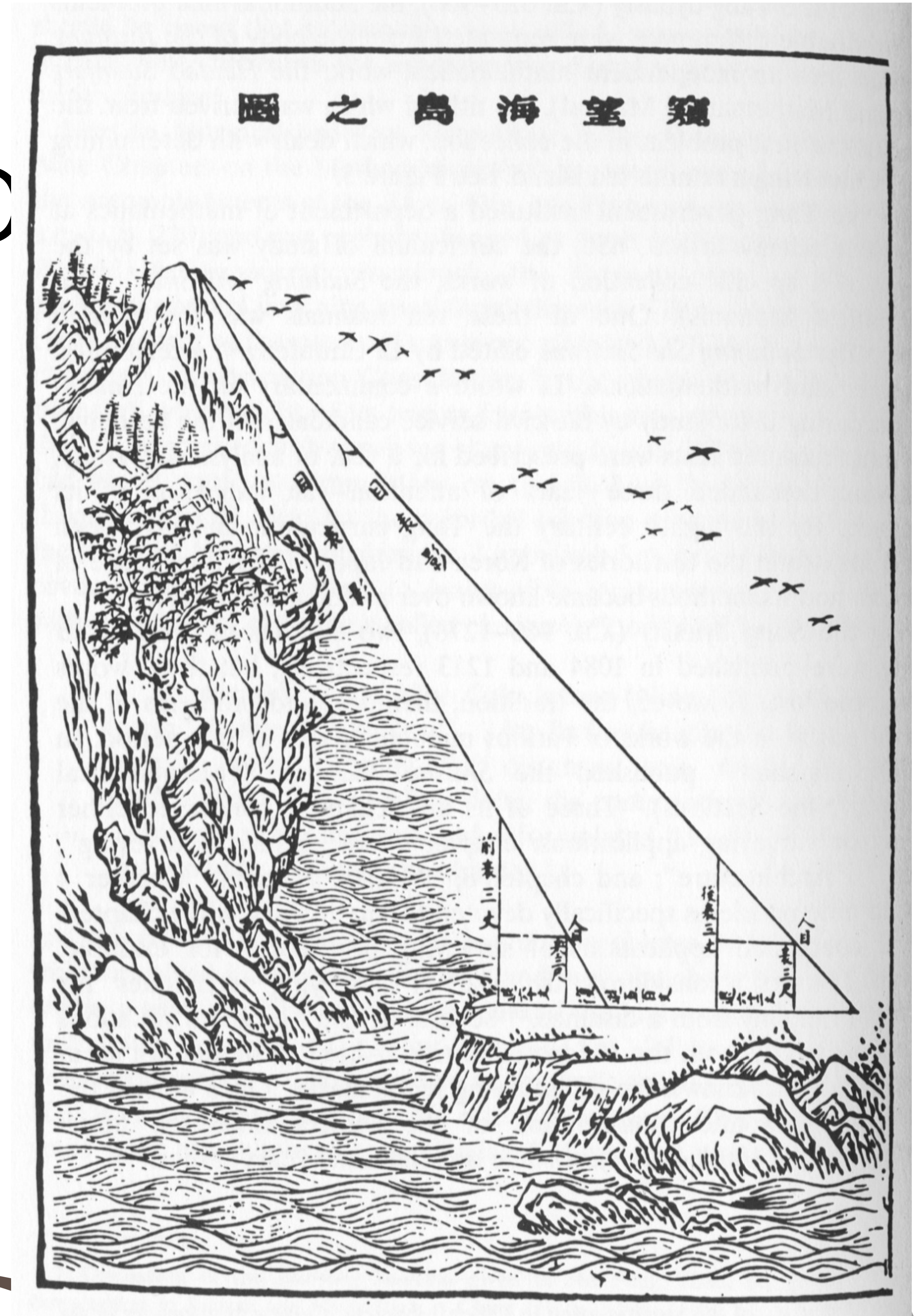
- Everything besides $\overline{AB}$ is known. So we can find $\overline{AB}$.



Similar Wo

- Similar work was done in China, as documented in Liu Hui's 263 AD book *Sea Island Mathematical Manual*, which was an extension of the final chapter in his edition of the *Nine Chapters*.

- Example: Measuring the height of cliffs.



The People Sparking Geometry

- Egyptian surveyors may have sparked the Greeks' formal study of geometry.

- Proclus wrote that Thales "went to Egypt and thence introduced [Geometry] two the Greeks"

- Plato, in his dialogue *Timaeus*, wrote that Greeks are "always children" because they did not understand geometry like the Egyptians. In another dialogue called *Laws*, he wrote that Greek children "should learn as much of these subjects as the innumerable crowd of children in Egypt."

Math Justice For The People

In the words of one surveyor,

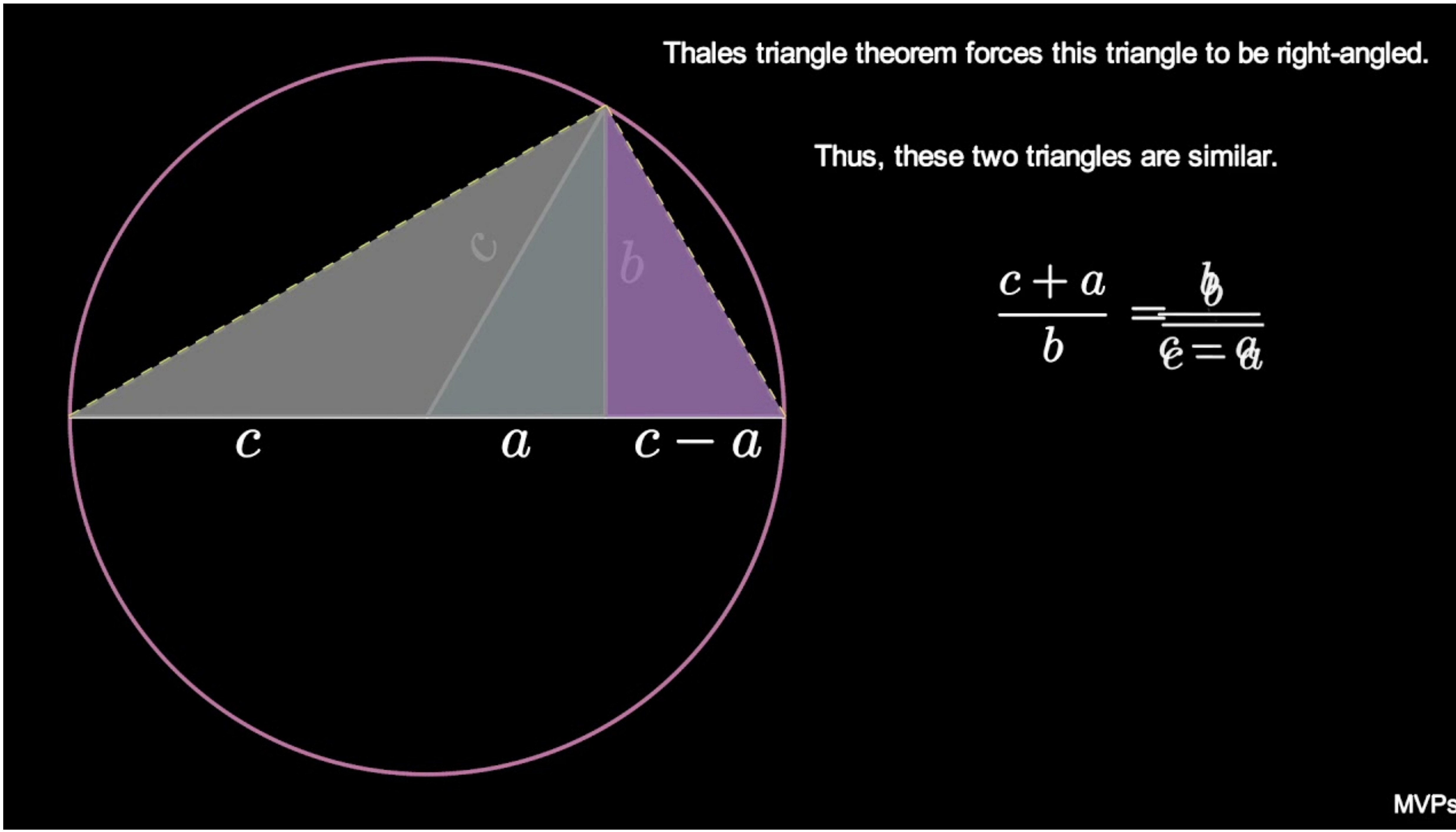
When I go to divide a plot, I can divide it; when I go to apportion a field, I can apportion the pieces; so that when wrong men have a quarrel, I soothe their hearts. [. . .] Brother will be at peace with brother.

- E.g., To decide how much land an Egyptian was owed, a subjective bureaucrat was replaced with an objective calculation.

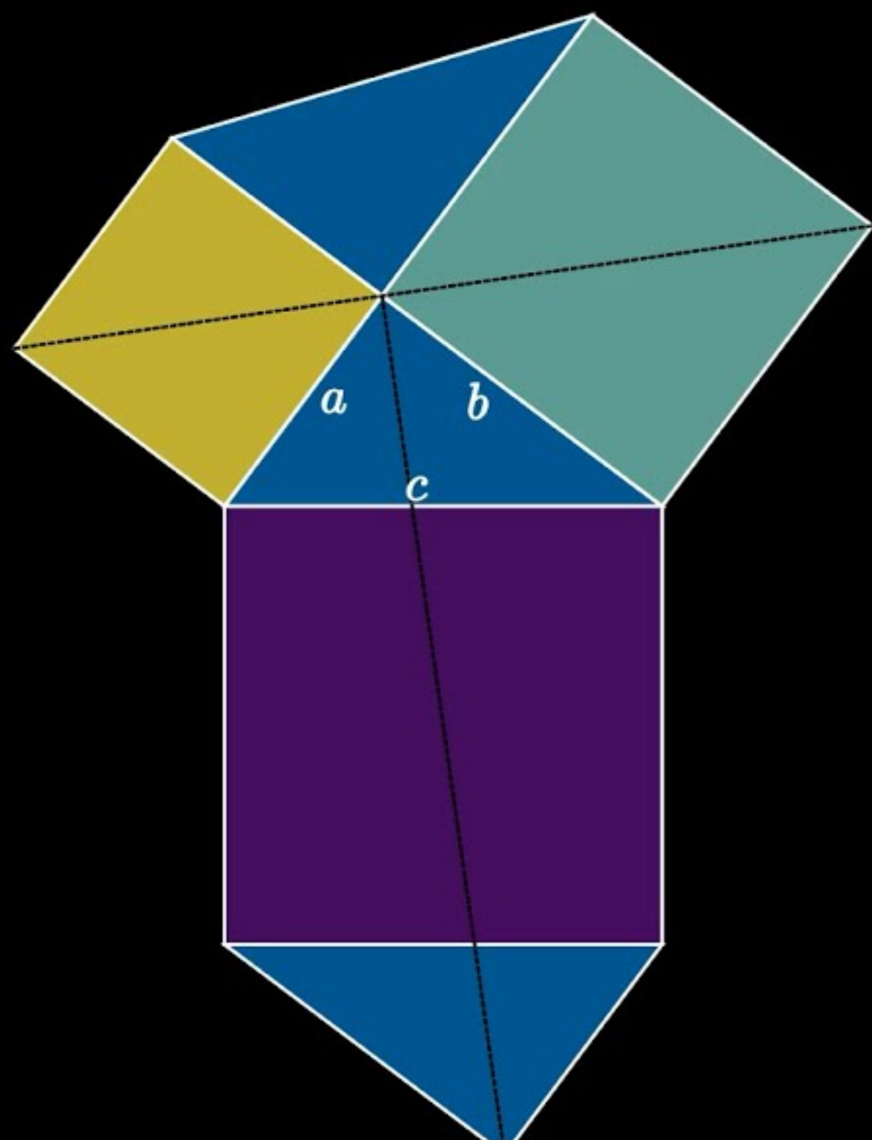
- Indeed, the Sumerian word for *justice* means *straightness, equality, squareness*. Today, Lady Justice is blindfolded, holding scales. In the earliest civilization, she was a geometer.

More Proofs of the Pythagorean Theorem

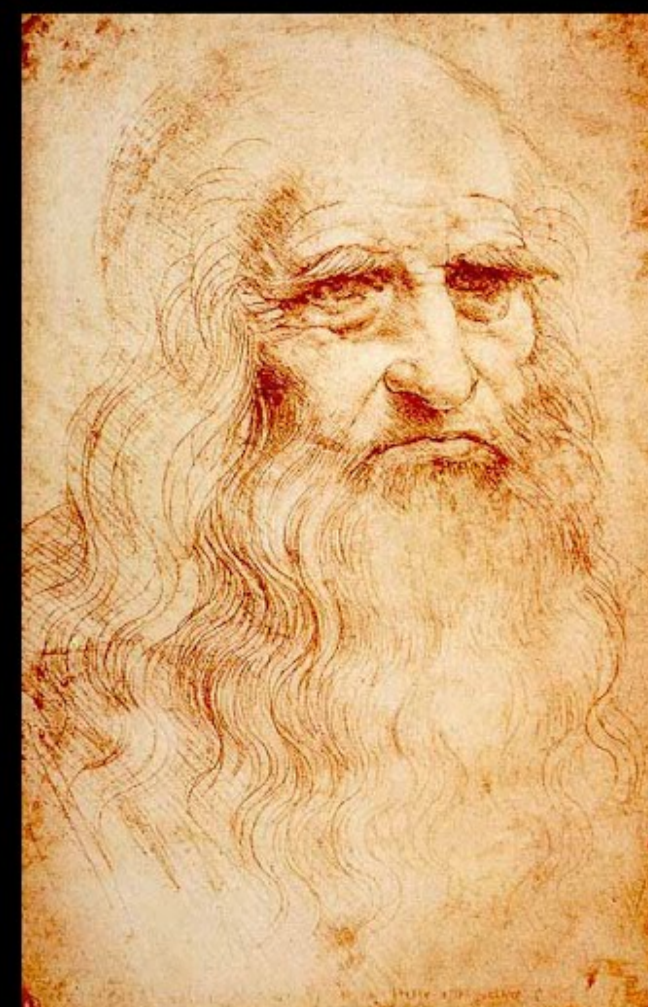
More Pythagorean Proofs



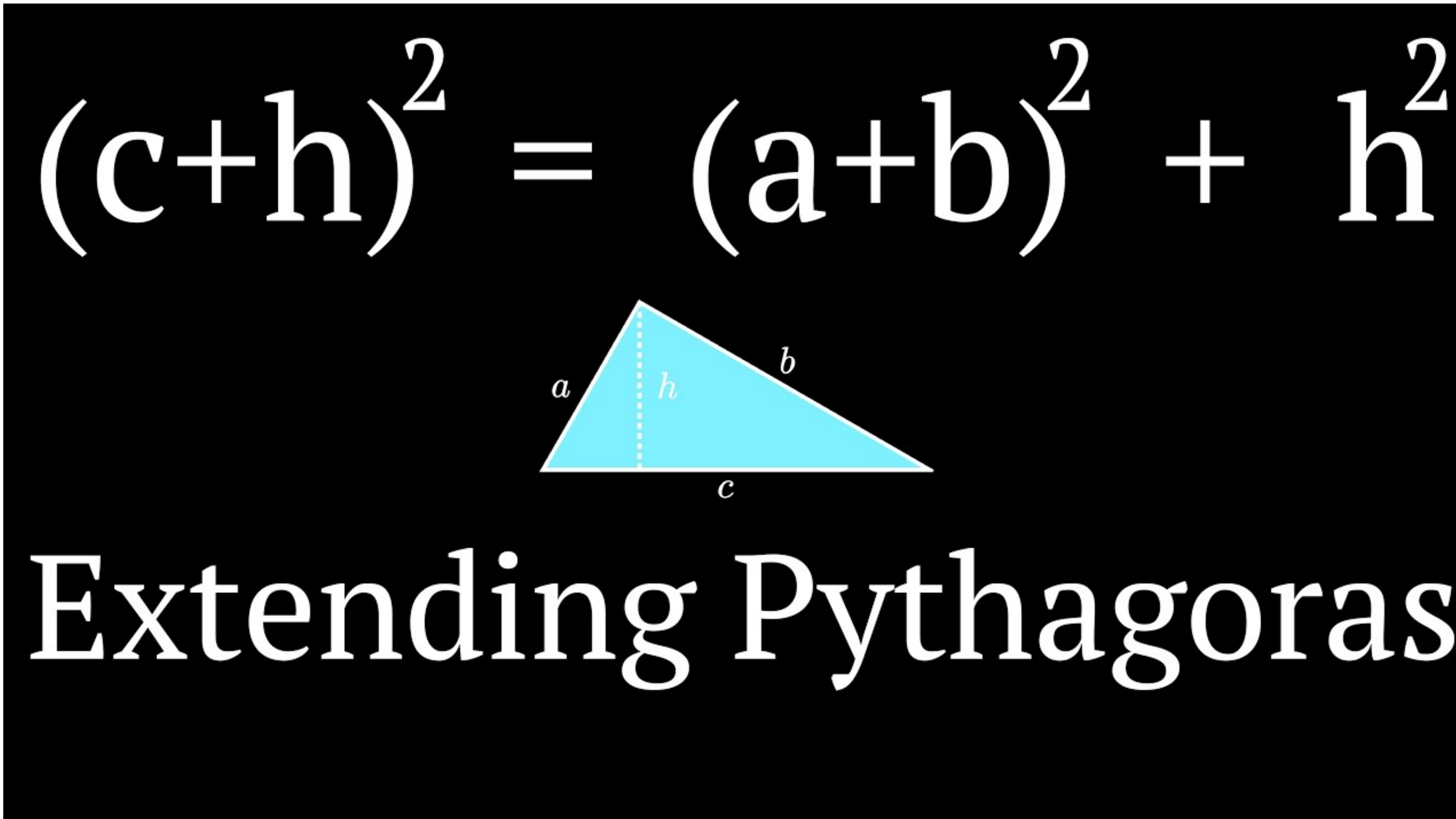
More Pythagorean Proofs



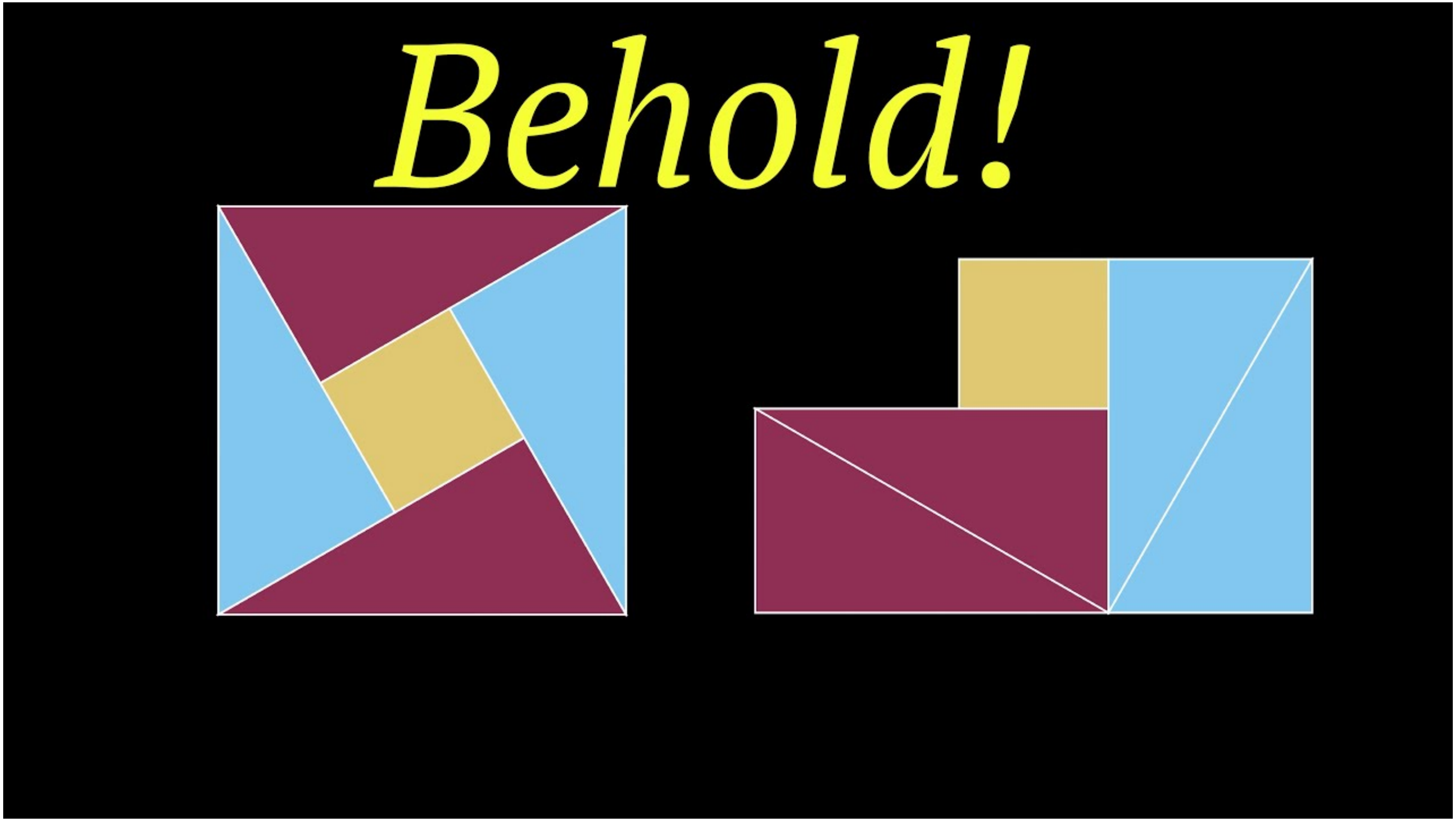
$$a^2+b^2=c^2$$



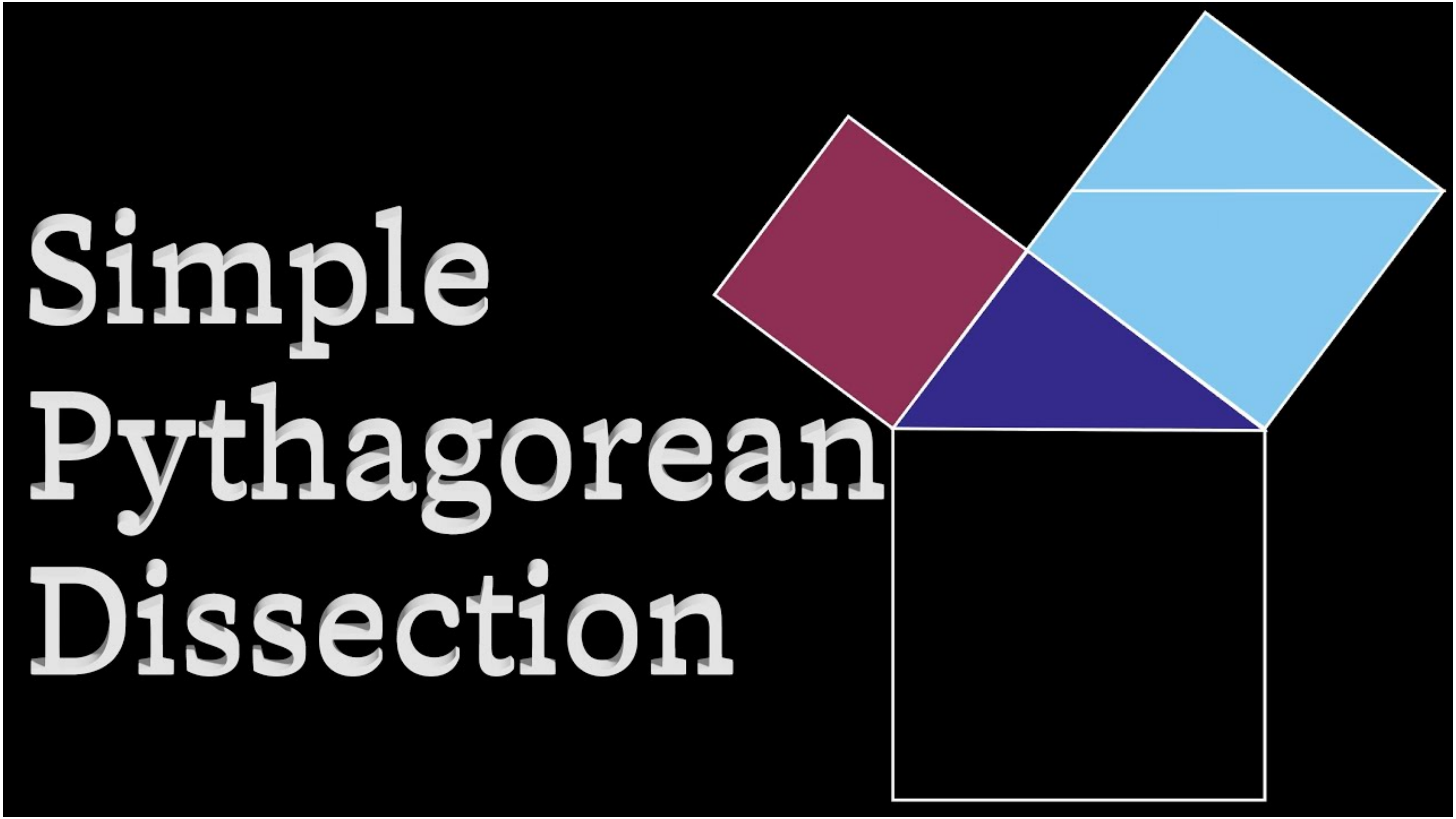
More Pythagorean Proofs



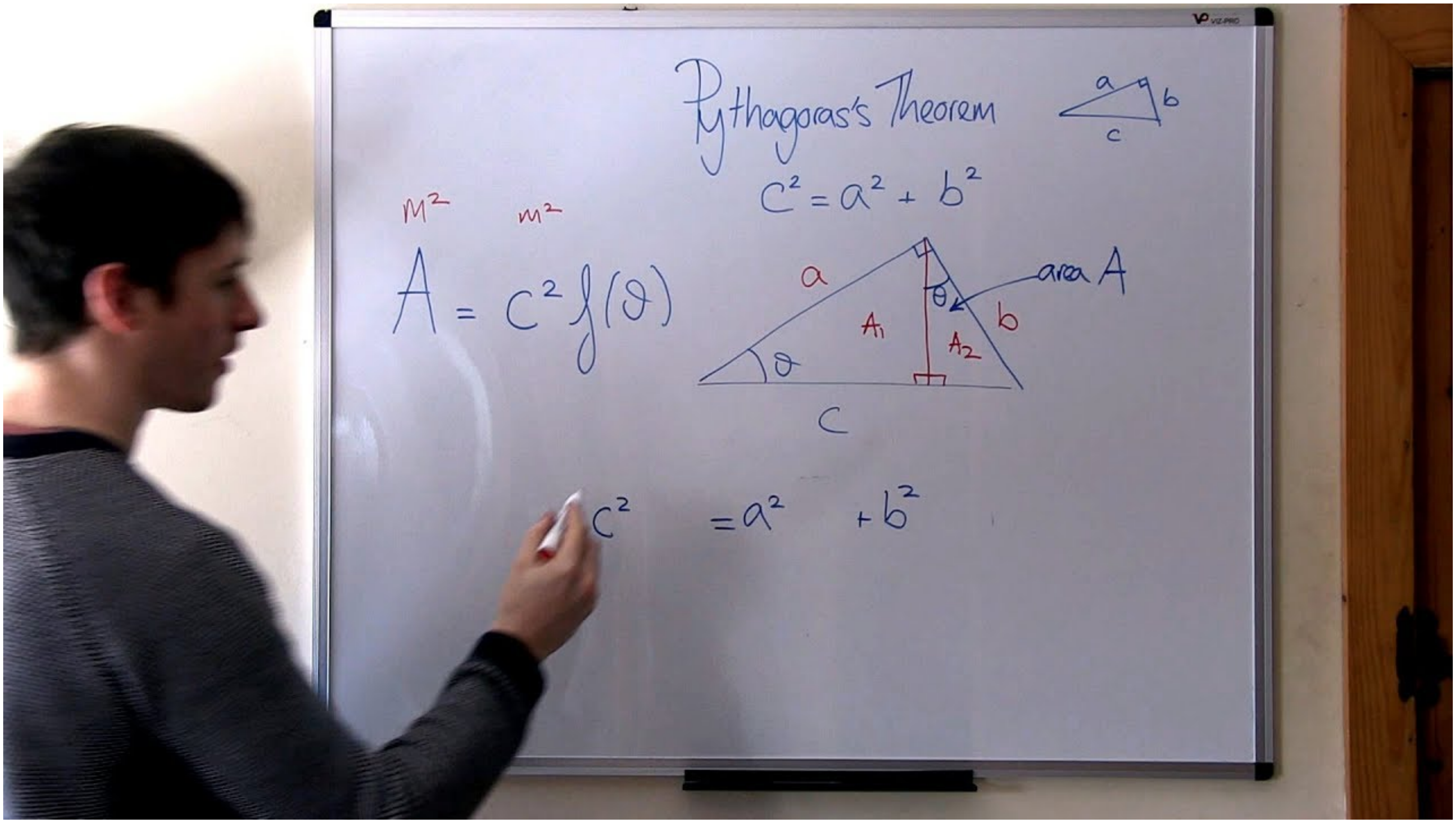
More Pythagorean Proofs



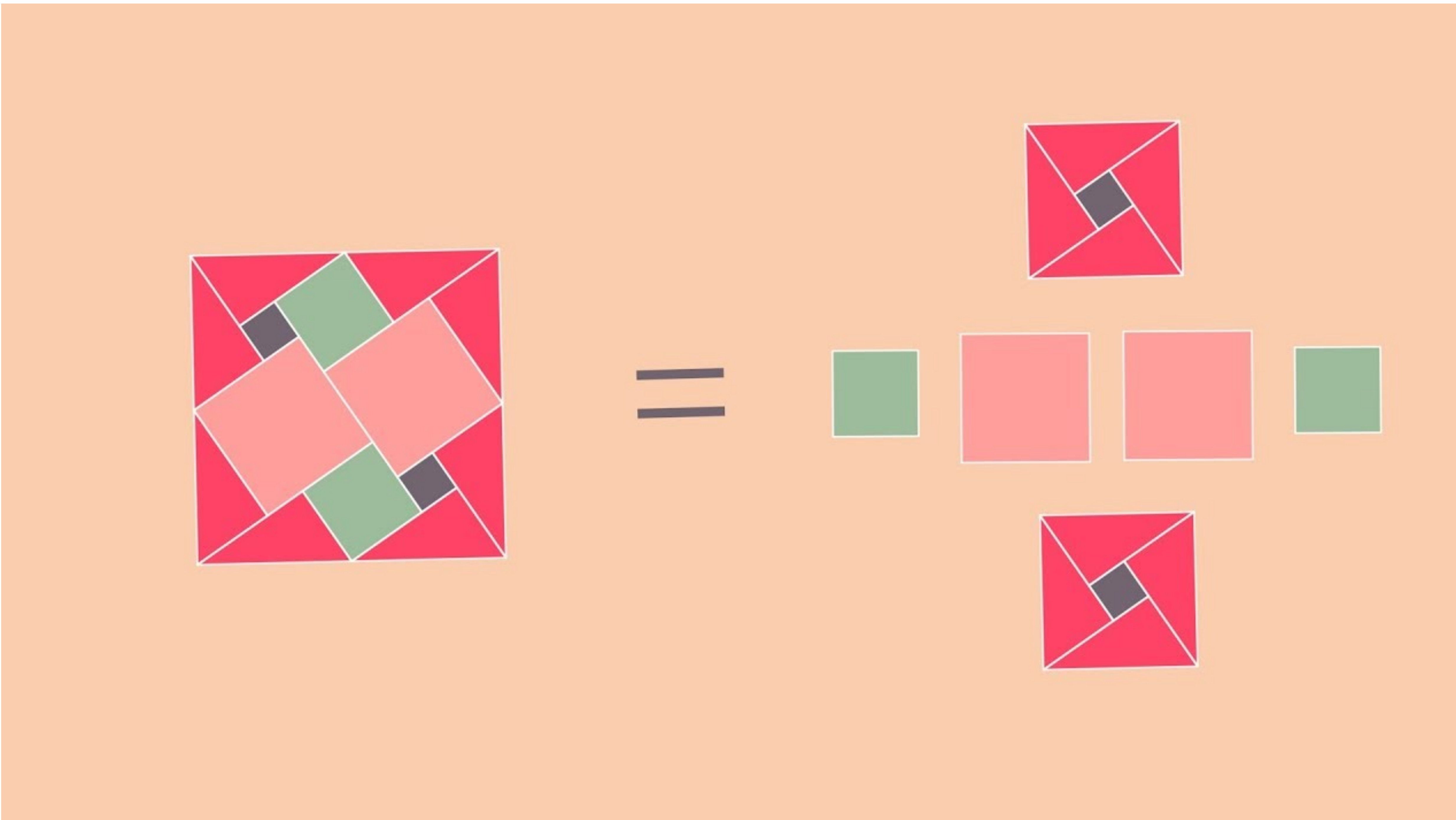
More Pythagorean Proofs



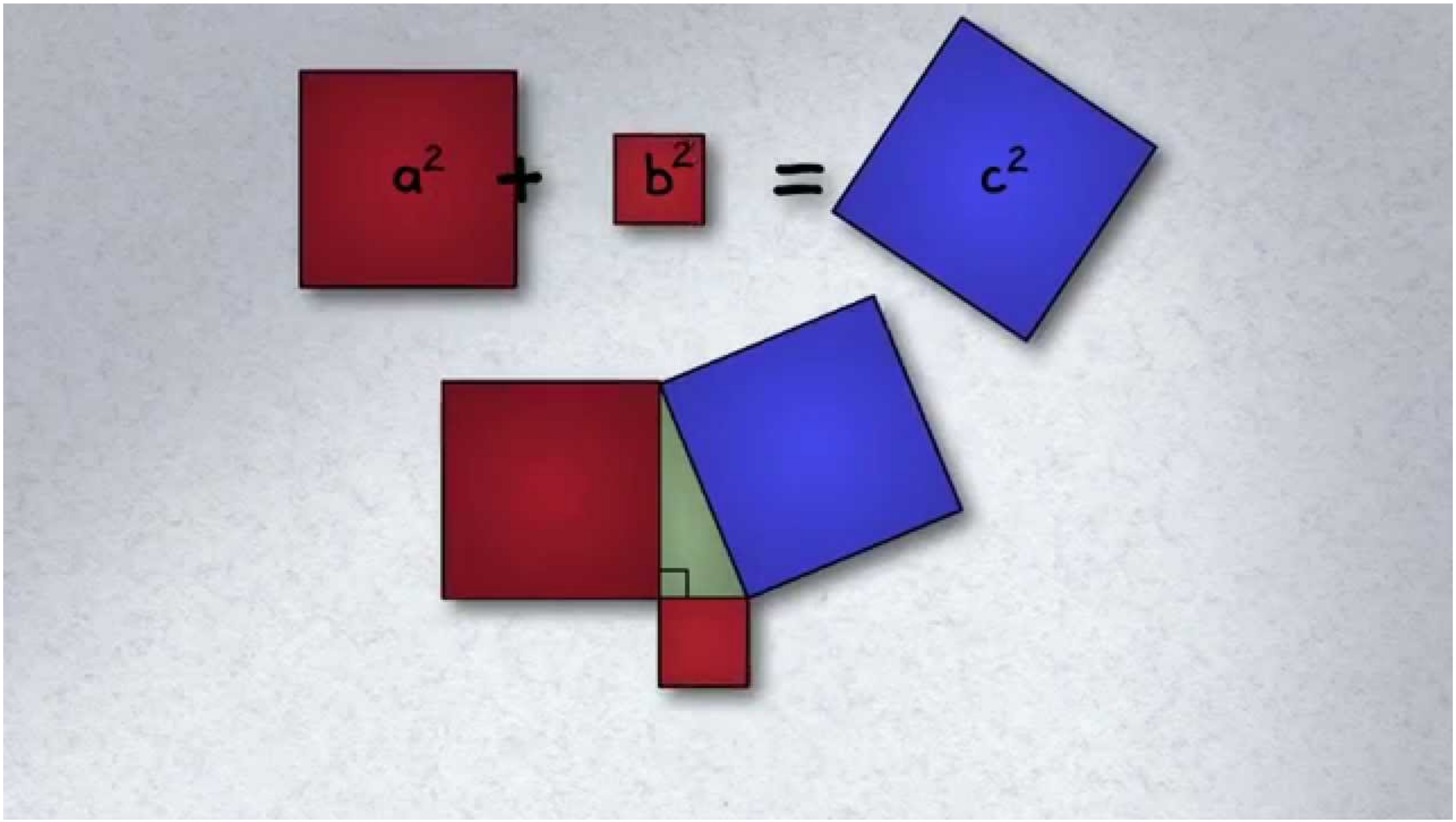
More Pythagorean Proofs



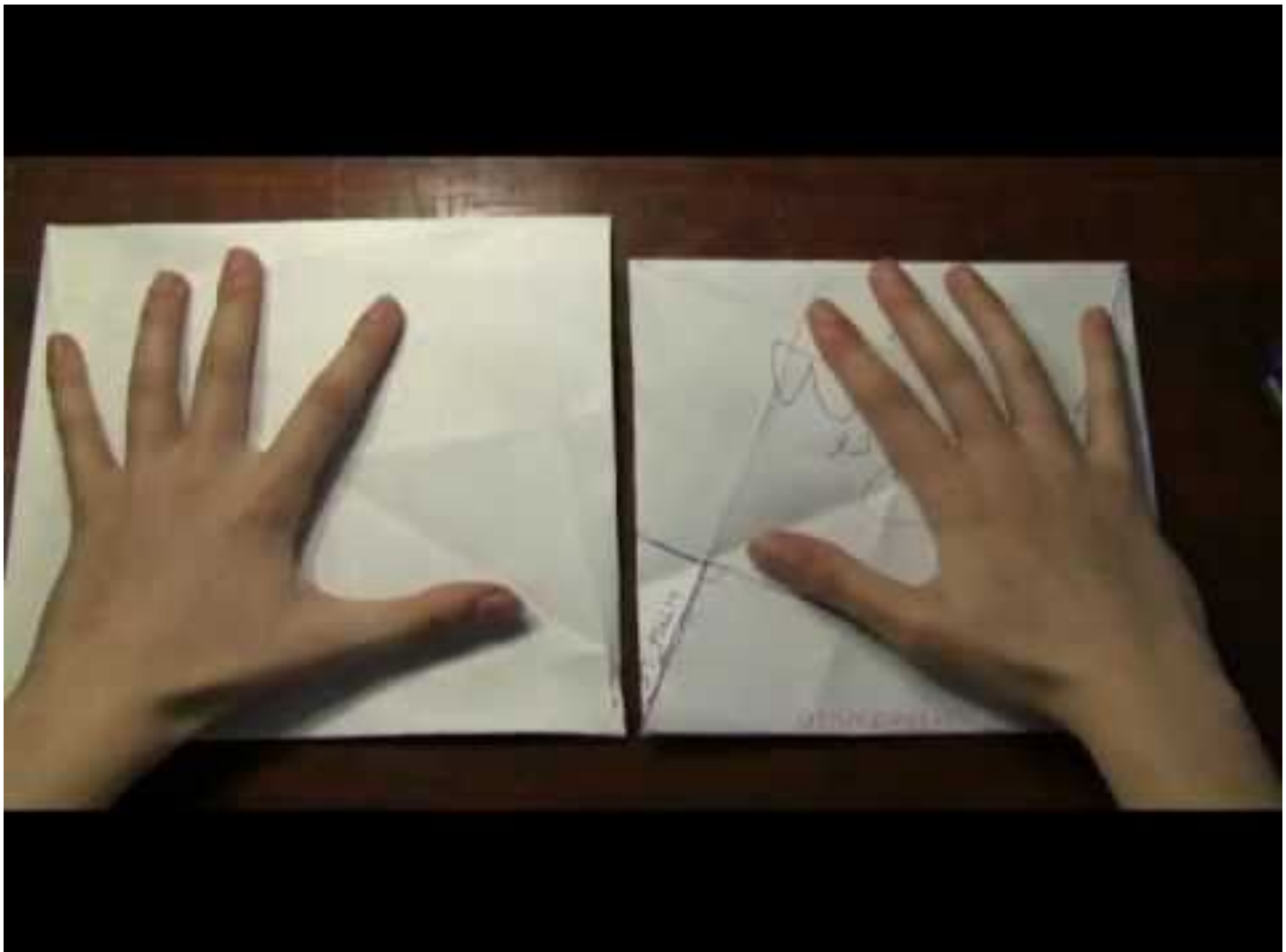
More Pythagorean Proofs



More Pythagorean Proofs



Pythagorean Theorem

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