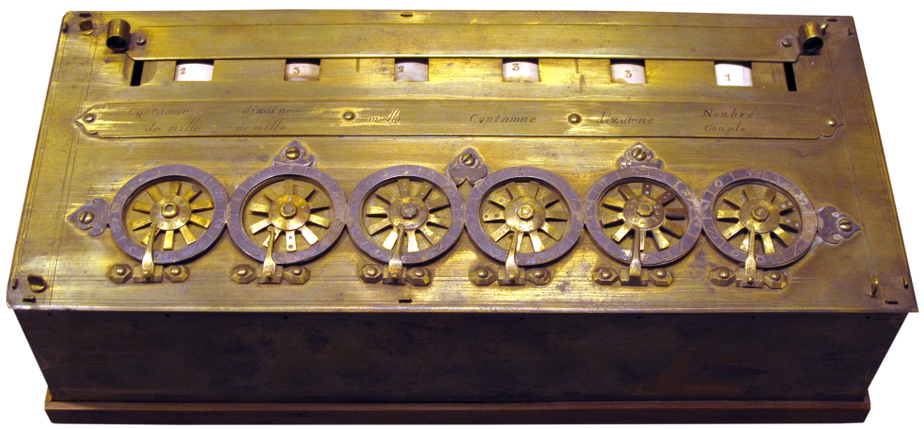
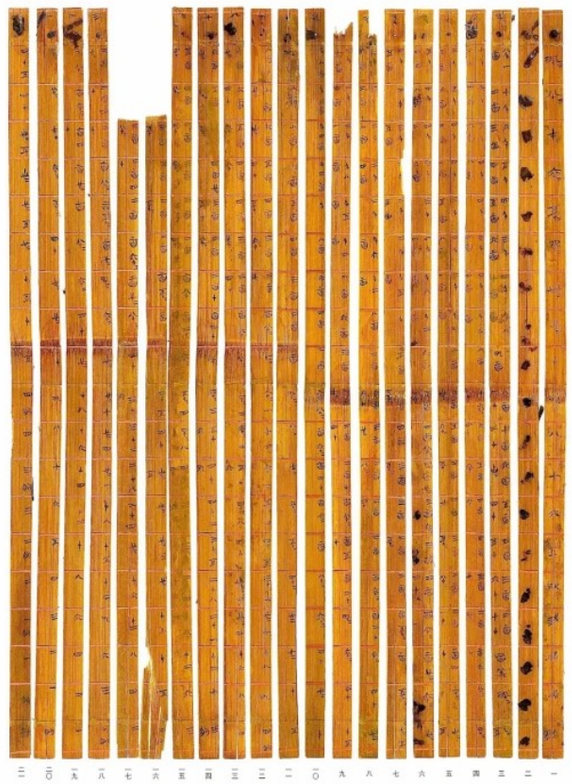
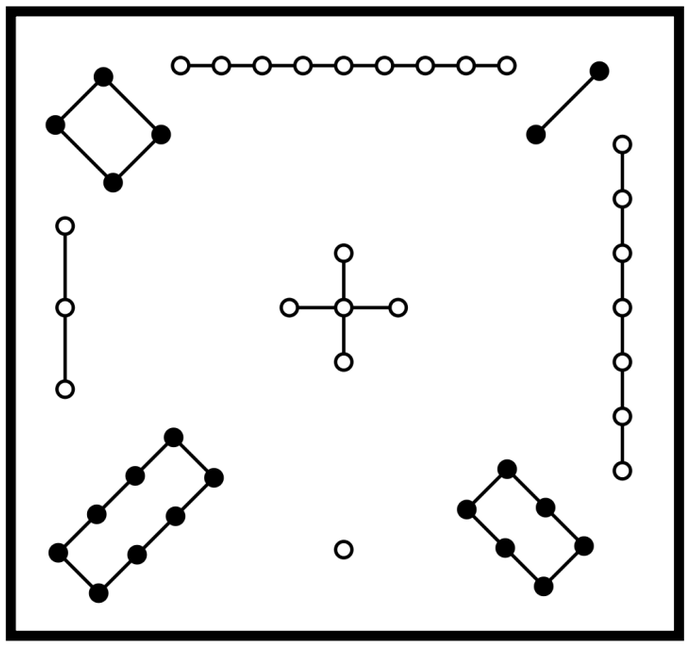
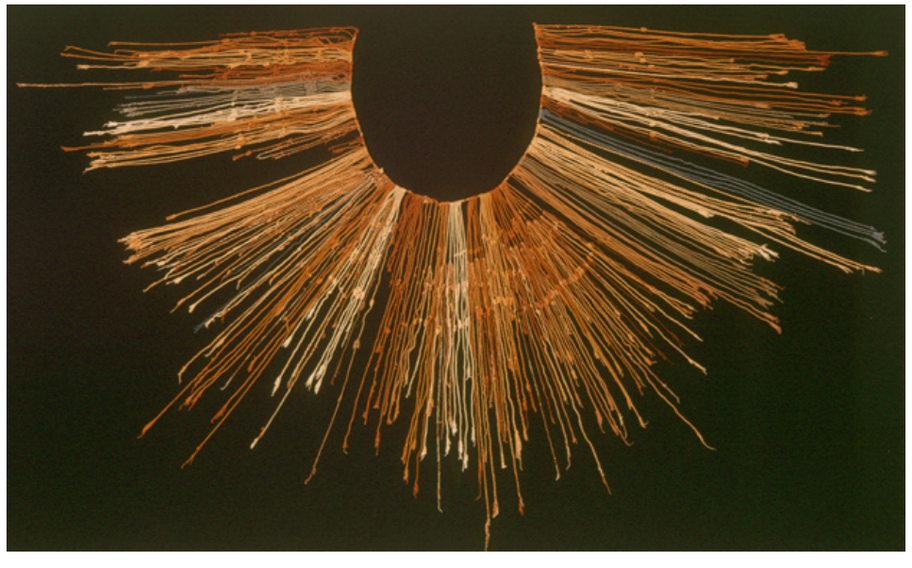






The History of Mathematics

Math 190, Professor Jay Cummings

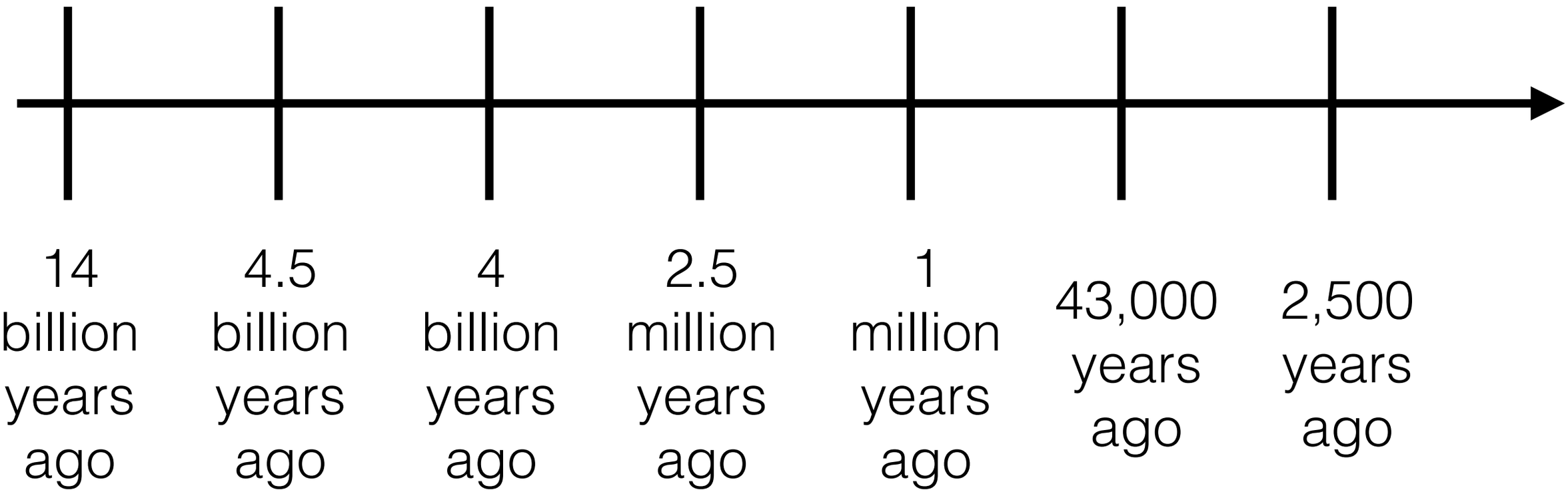


When Did Math Begin?

When Did Math Begin?

- Talk to a classmate about when you think the history of math began.

A Brief History of Time



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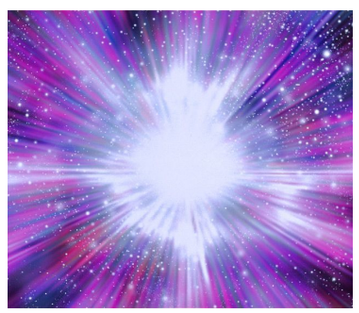

Universe born

14 billion years ago

4.5 billion years ago

4 billion years ago

2.5 million years ago

1 million years ago

43,000 years ago

2,500 years ago

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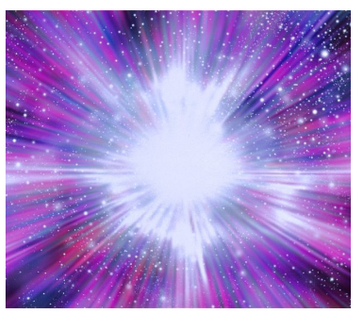
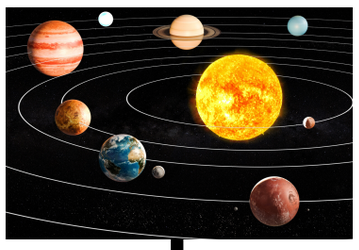


Universe

Solar system born



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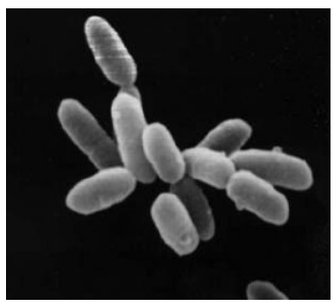
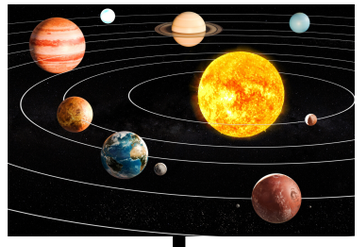


Life on Earth



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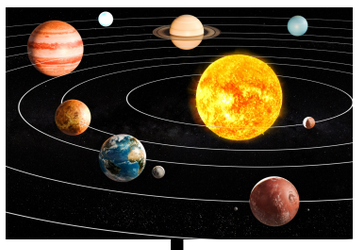

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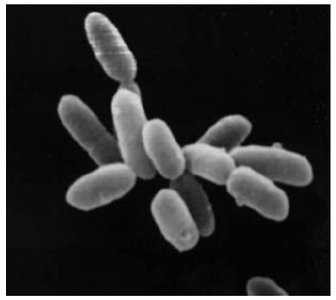

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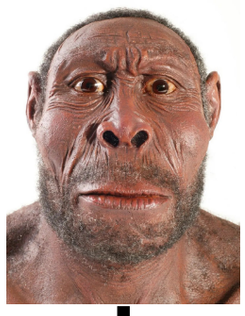

Homo Habilis tools

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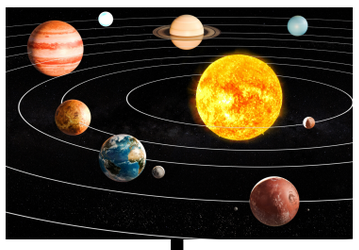

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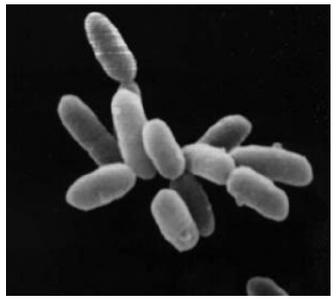

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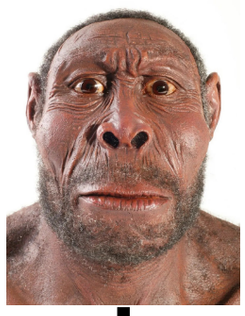

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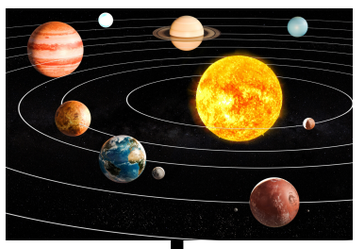

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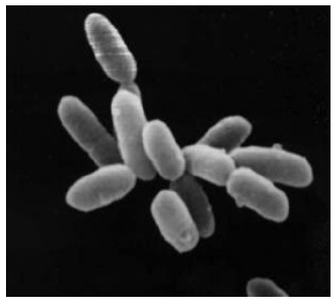

Life on Earth

4 billion years ago

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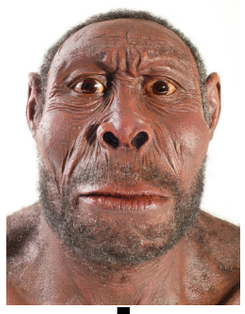

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Tally marks on bone

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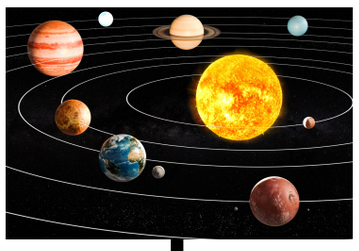

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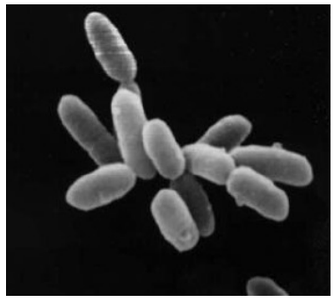

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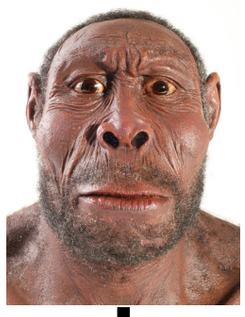

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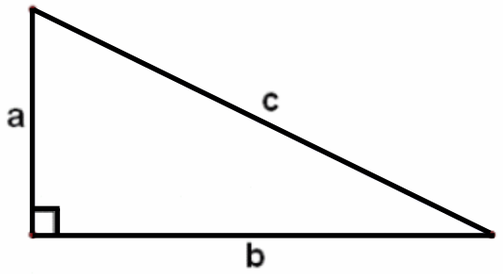

Pythagorean theorem proved

2,500 years ago

Lebombo Bone

Lebombo Bone

- From ~41,000 BC.
- 29 clear notches.

Lebombo Bone

Ishango Bone

Ishango Bone

- From ~18,000 BC.

Ishango Bone

- From ~18,000 BC.
- Has 3 columns, with 48, 60 and 60 notches.

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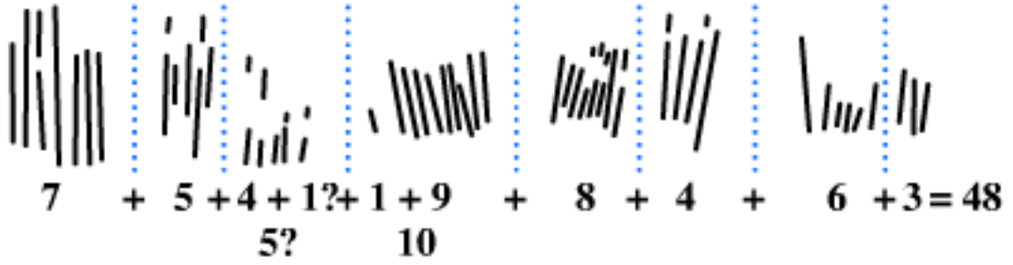

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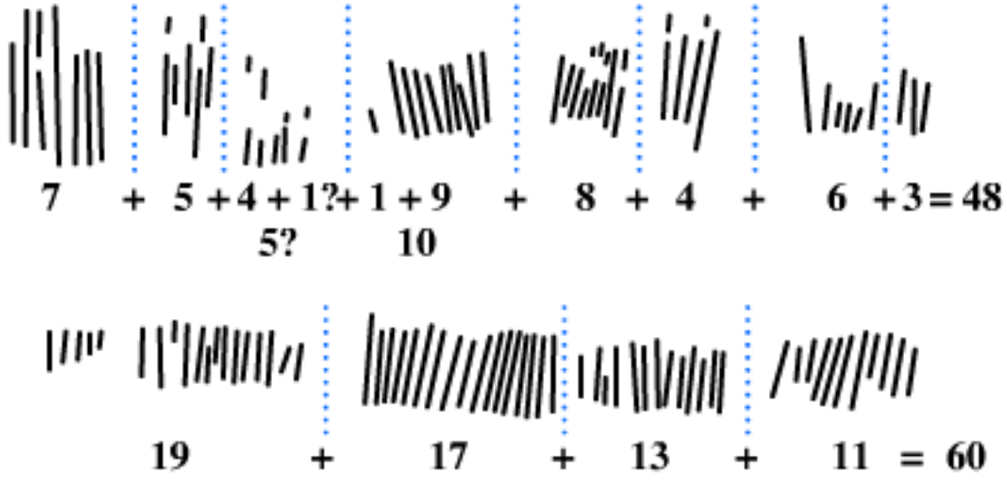

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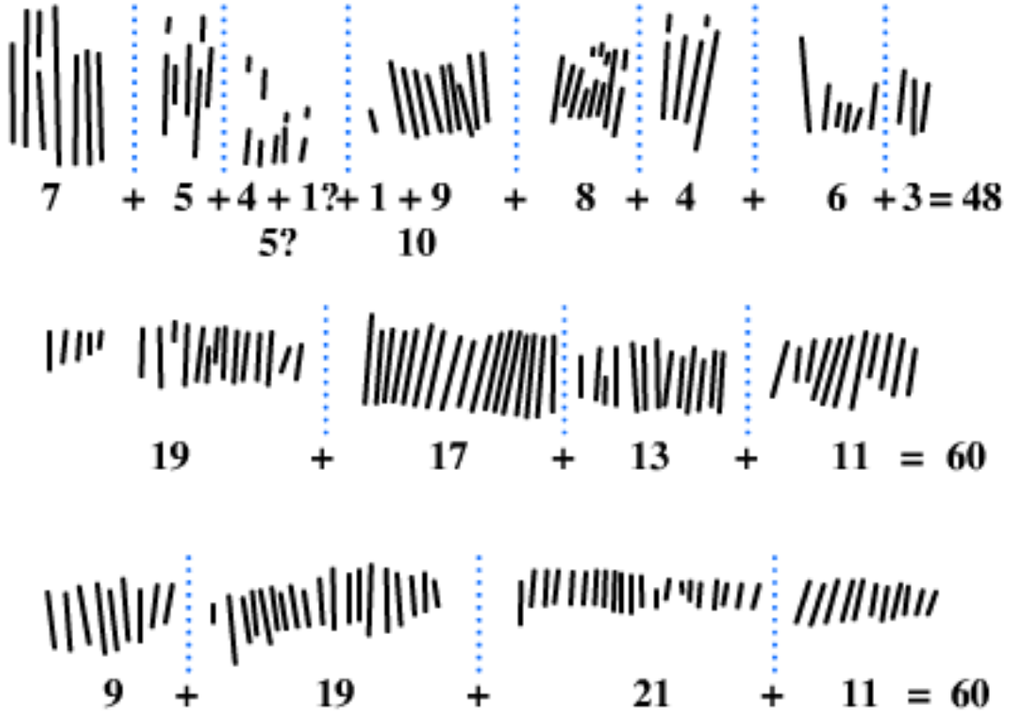

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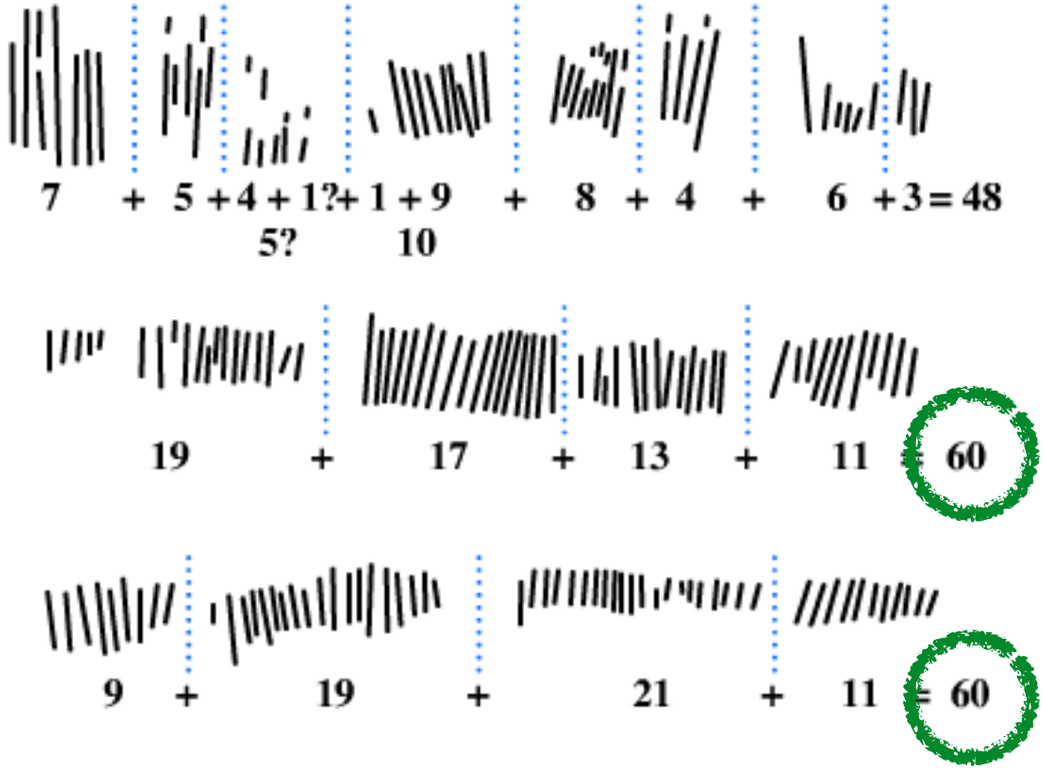

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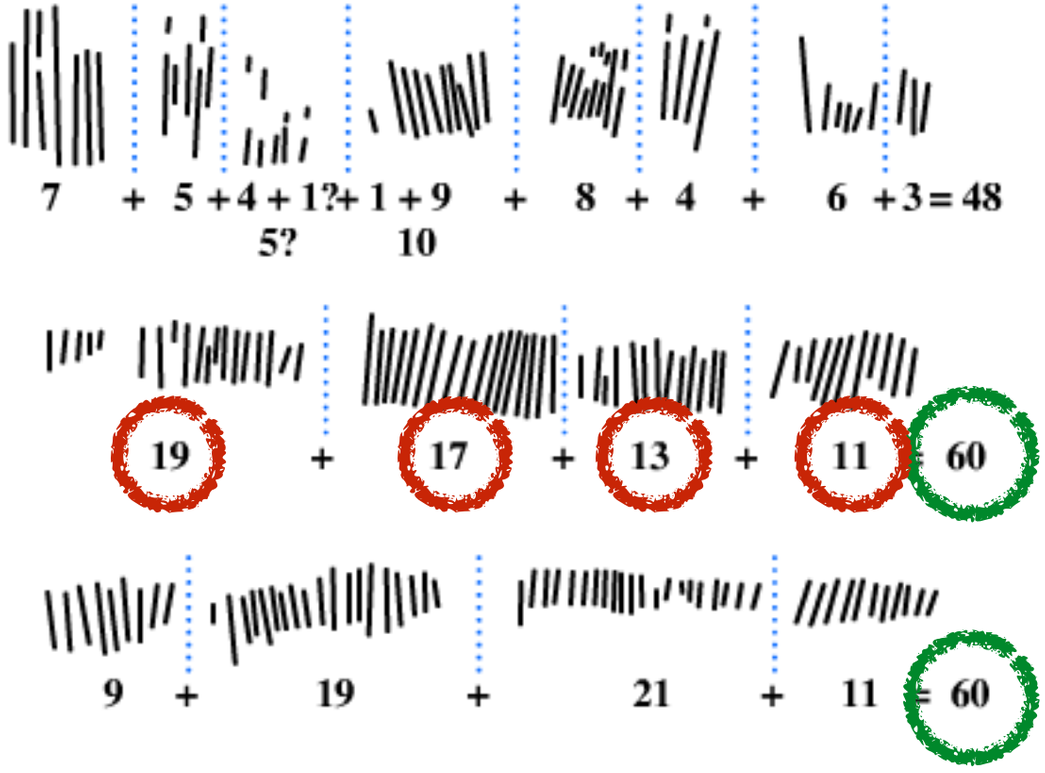

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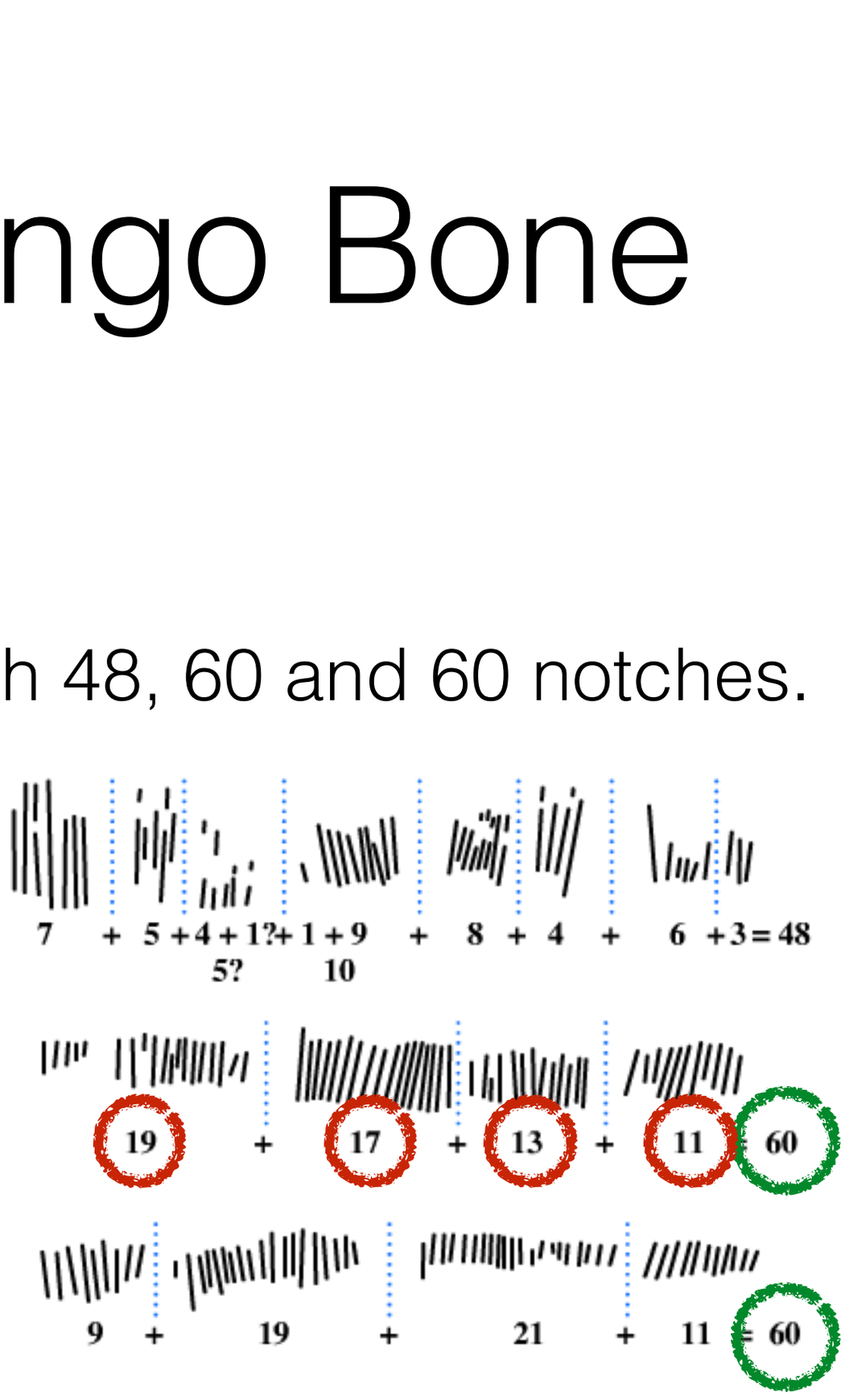

Both bones are the fibula of baboons!

Ishango Bone

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- We will start with ancient counting by humans, and go all the way to the 21st century.

- First three chapters discuss ancient math.

- Last seven chapters begin with an early idea, and track its evolution to modern theorems and major fields of study. This creates a natural balance between eras.

How Humans Count

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Number Bases

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where each a_i is an integer between 0 and $b - 1$.

Converting the number $(a_k a_{k-1} \ldots a_2 a_1 a_0)_b$ to base 10 gives

$$a_k b^k + a_{k-1} b^{k-1} + \ldots + a_2 b^2 + a_1 b^1 + a_0 \cdot b^0.$$

Practice

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Example. $(123)_4$ is a base 4 number.

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We don't need any 9s. After 2 copies of 3, we are left with

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- 2. Multiplicative Grouping System: Have a character for each power of 10

  Example: Chinese-Japanese numeral system

4 Ways People Write Numbers

$1 = 一$	$10 = 十$	$100 = 一百$	$1{,}000 = 一千$
$2 = 二$	$20 = 二十$	$200 = 二百$	$2{,}000 = 二千$
$3 = 三$	$30 = 三十$	$300 = 三百$	$3{,}000 = 三千$
$4 = 四$	$40 = 四十$	$400 = 四百$	$4{,}000 = 四千$
$5 = 五$	$50 = 五十$	$500 = 五百$	$5{,}000 = 五千$
$6 = 六$	$60 = 六十$	$600 = 六百$	$6{,}000 = 六千$
$7 = 七$	$70 = 七十$	$700 = 七百$	$7{,}000 = 七千$
$8 = 八$	$80 = 八十$	$800 = 八百$	$8{,}000 = 八千$
$9 = 九$	$90 = 九十$	$900 = 九百$	$9{,}000 = 九千$
$10 = 十$	$100 = 一百$	$1{,}000 = 一千$	$10{,}000 = 一万$

Example: Chinese-Japanese numeral system

4 Ways People Write Numbers

$1 = 一$	$10 = 十$	$100 = 一百$	$1{,}000 = 一千$
$2 = 二$	$20 = 二十$	$200 = 二百$	$2{,}000 = 二千$
$3 = 三$	$30 = 三十$	$300 = 三百$	$3{,}000 = 三千$
$4 = 四$	$40 = 四十$	$400 = 四百$	$4{,}000 = 四千$
$5 = 五$	$50 = 五十$	$500 = 五百$	$5{,}000 = 五千$
$6 = 六$	$60 = 六十$	$600 = 六百$	$6{,}000 = 六千$
$7 = 七$	$70 = 七十$	$700 = 七百$	$7{,}000 = 七千$
$8 = 八$	$80 = 八十$	$800 = 八百$	$8{,}000 = 八千$
$9 = 九$	$90 = 九十$	$900 = 九百$	$9{,}000 = 九千$
$10 = 十$	$100 = 一百$	$1{,}000 = 一千$	$10{,}000 = 一万$

Example: Chinese-Japanese numeral system

$$5,062 \;=\; 5 \cdot 1,000 + 6 \cdot 10 + 2$$

4 Ways People Write Numbers

$1 = 一$	$10 = 十$	$100 = 一百$	$1{,}000 = 一千$
$2 = 二$	$20 = 二十$	$200 = 二百$	$2{,}000 = 二千$
$3 = 三$	$30 = 三十$	$300 = 三百$	$3{,}000 = 三千$
$4 = 四$	$40 = 四十$	$400 = 四百$	$4{,}000 = 四千$
$5 = 五$	$50 = 五十$	$500 = 五百$	$5{,}000 = 五千$
$6 = 六$	$60 = 六十$	$600 = 六百$	$6{,}000 = 六千$
$7 = 七$	$70 = 七十$	$700 = 七百$	$7{,}000 = 七千$
$8 = 八$	$80 = 八十$	$800 = 八百$	$8{,}000 = 八千$
$9 = 九$	$90 = 九十$	$900 = 九百$	$9{,}000 = 九千$
$10 = 十$	$100 = 一百$	$1{,}000 = 一千$	$10{,}000 = 一万$

Example: Chinese-Japanese numeral system

$$5,062 \;=\; 5 \cdot 1,000 + 6 \cdot 10 + 2$$

$$= 五千六十二$$

4 Ways People Write Numbers

4 Ways People Write Numbers

- 3. Simple Grouping System: Add everything up.

4 Ways People Write Numbers

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  Example: Egyptian hieroglyphs

4 Ways People Write Numbers

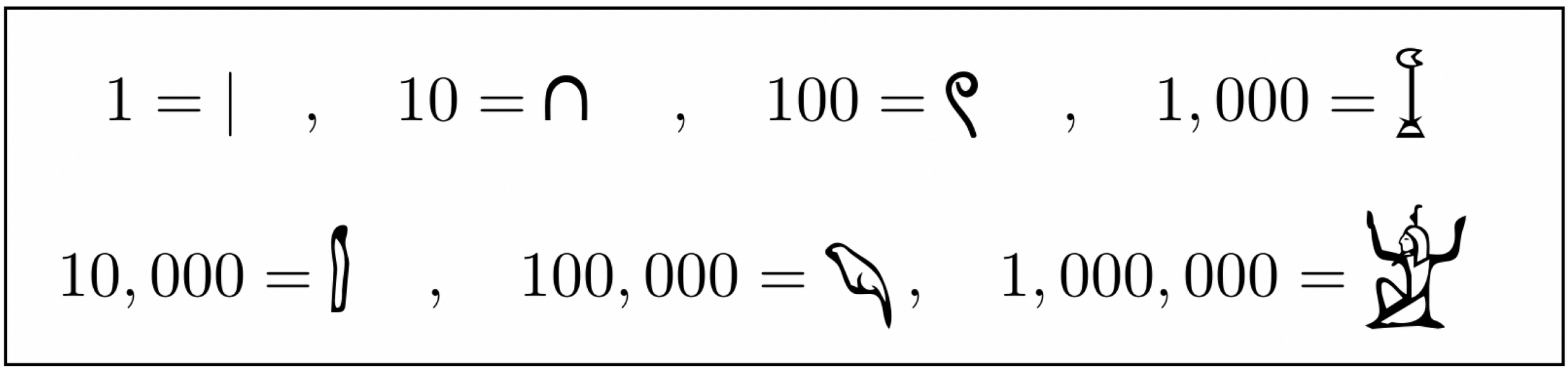
- 3. Simple Grouping System: Add everything up.

$1 =$, $10 =$, $100 =$, $1,000 =$ 

$10,000 =$  , $100,000 =$  , $1,000,000 =$ 

4 Ways People Write Numbers

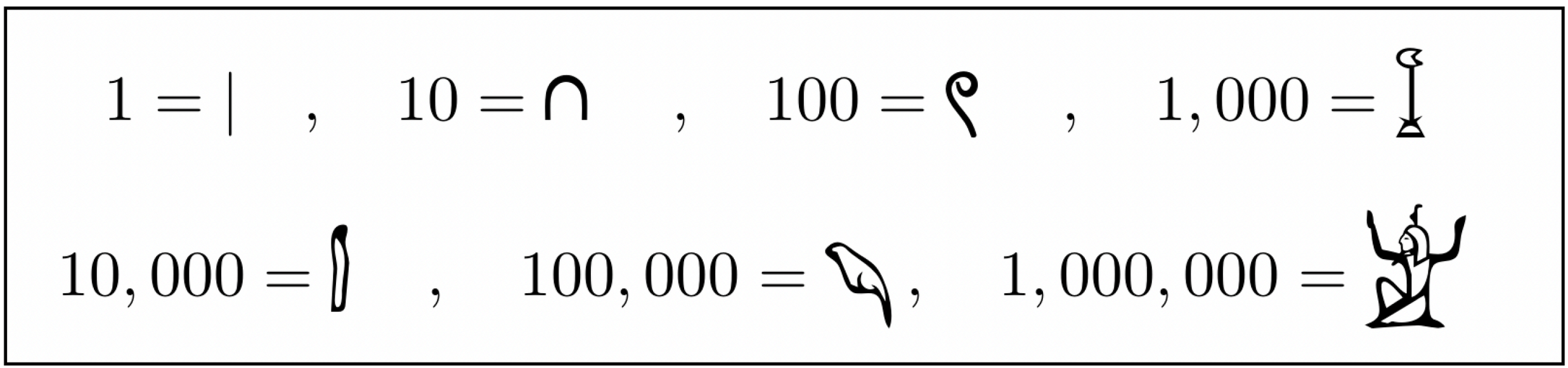
- 3. Simple Grouping System: Add everything up.



321

4 Ways People Write Numbers

- 3. Simple Grouping System: Add everything up.



4 Ways People Write Numbers

- 3. Simple Grouping System: Add everything up.

$1 =$, $10 =$, $100 =$, $1,000 =$ 

$10,000 =$  , $100,000 =$  , $1,000,000 =$ 

$321 =$ 

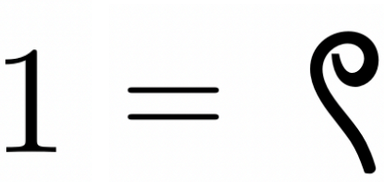
$=$ 

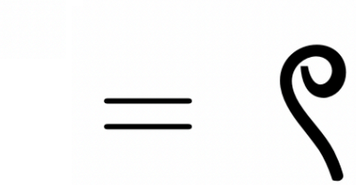
4 Ways People Write Numbers

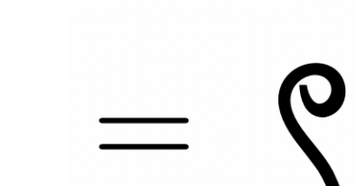
- 3. Simple Grouping System: Add everything up.

$1 =$, $10 =$, $100 =$, $1,000 =$

$10,000 =$  , $100,000 =$  , $1,000,000 =$

$321 =$ 

$=$ 

$=$ 

4 Ways People Write Numbers

- 3. Simple Grouping System: Add everything up.

$1 =$  , $10 =$, $100 =$, $1,000 =$ 

$10,000 =$  , $100,000 =$  , $1,000,000 =$ 

$321 =$ 

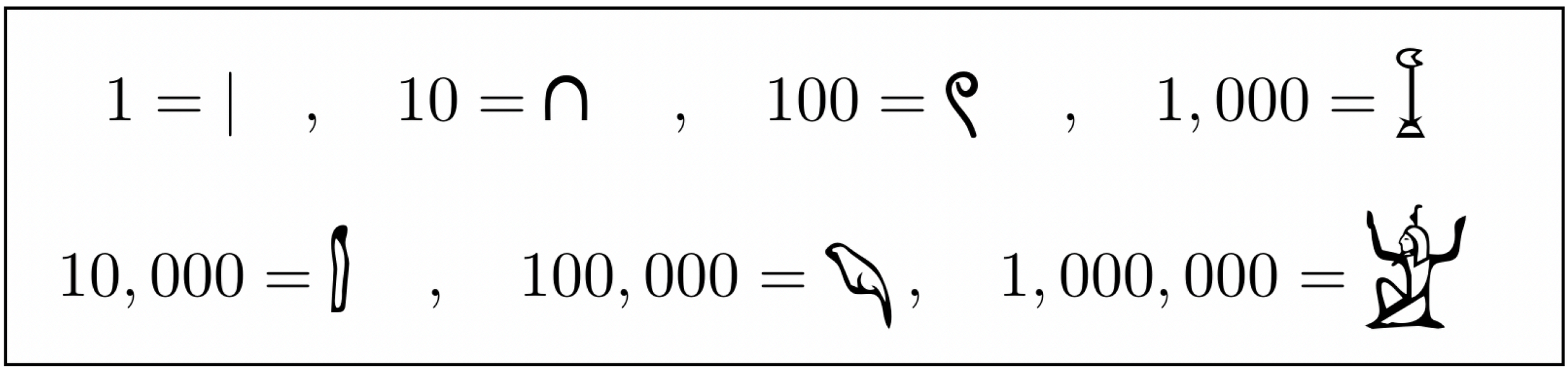
$=$ 

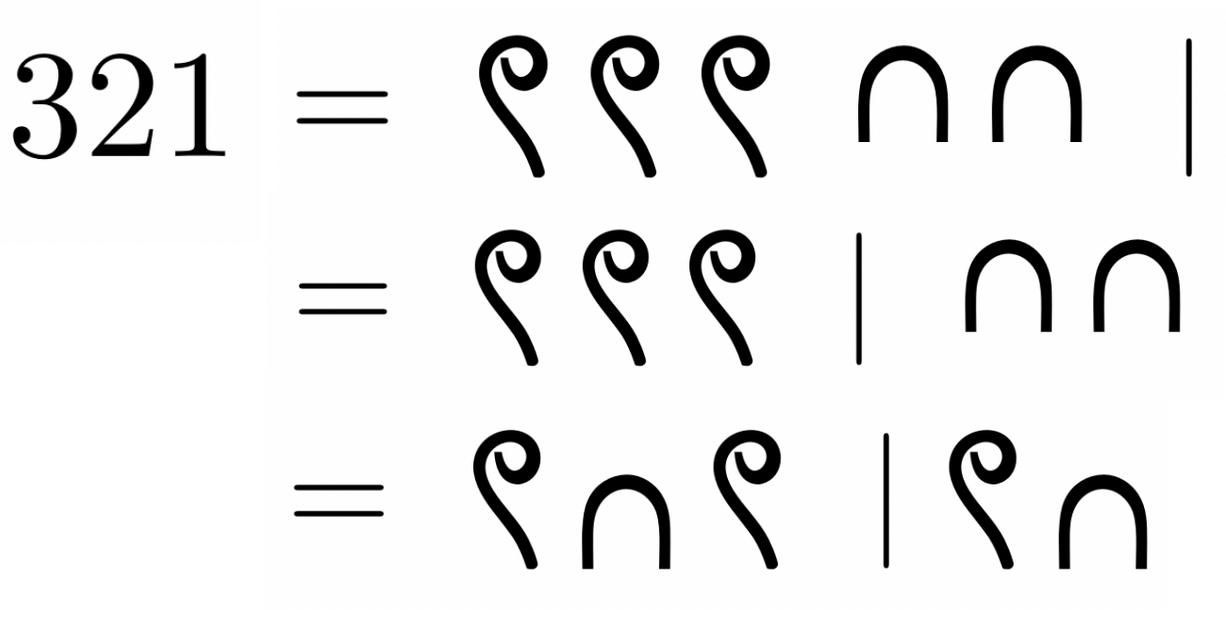
$=$ 

- Another example: Roman numerals.

4 Ways People Write Numbers

- 3. Simple Grouping System: Add everything up.





- Another example: Roman numerals.
- Except for the more modern subtraction rules

4 Ways People Write Numbers

4 Ways People Write Numbers

- 4. Ciphered systems: It's… complicated. For a base b, we need symbols for

4 Ways People Write Numbers

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$$1,2,\ldots,b-1$$
$$b,2b,\ldots,(b-1)b$$
$$b^2,2b^2,\ldots,(b-1)b^2$$
$$b^3,2b^3,\ldots,(b-1)b^3$$
$$\vdots$$

4 Ways People Write Numbers

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$$1,2,\ldots,b-1$$
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$$\vdots$$

Example: Ionic Greek system

4 Ways People Write Numbers

$1 = \alpha$ (alpha)	$10 = \iota$ (iota)	$100 = \rho$ (rho)
$2 = \beta$ (beta)	$20 = \kappa$ (kappa)	$200 = \sigma$ (sigma)
$3 = \gamma$ (gamma)	$30 = \lambda$ (lambda)	$300 = \tau$ (tau)
$4 = \delta$ (delta)	$40 = \mu$ (mu)	$400 = \upsilon$ (upsilon)
$5 = \varepsilon$ (epsilon)	$50 = \nu$ (nu)	$500 = \phi$ (phi)
$6 = \varsigma$ (vau)	$60 = \xi$ (xi)	$600 = \chi$ (chi)
$7 = \zeta$ (zeta)	$70 = o$ (omicron)	$700 = \psi$ (psi)
$8 = \eta$ (eta)	$80 = \pi$ (pi)	$800 = \omega$ (omega)
$9 = \theta$ (theta)	$90 = \varsigma$ (koppa)	$900 = \curlyvee$ (sampi)

Example: Ionic Greek system

Question: Which system is best?

Zero

Zero

The Indians and the Mayans were the two civilizations to discover zero and recognize its deep potential as:

Zero

The Indians and the Mayans were the two civilizations to discover zero and recognize its deep potential as:

- A symbol

Zero

The Indians and the Mayans were the two civilizations to discover zero and recognize its deep potential as:

- A symbol
- A number

Zero

The Indians and the Mayans were the two civilizations to discover zero and recognize its deep potential as:

- A symbol
- A number
- A magnitude

Zero

The Indians and the Mayans were the two civilizations to discover zero and recognize its deep potential as:

- A symbol
- A number
- A magnitude
- A direction separator

Zero

The Indians and the Mayans were the two civilizations to discover zero and recognize its deep potential as:

- A symbol
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- A magnitude
- A direction separator
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(This is a big deal)

Zero

The Indians and the Mayans were the two civilizations to discover zero and recognize its deep potential as:

- A symbol
- A number
- A magnitude
- A direction separator
- A place-holder
- An idea

(This is a big deal)

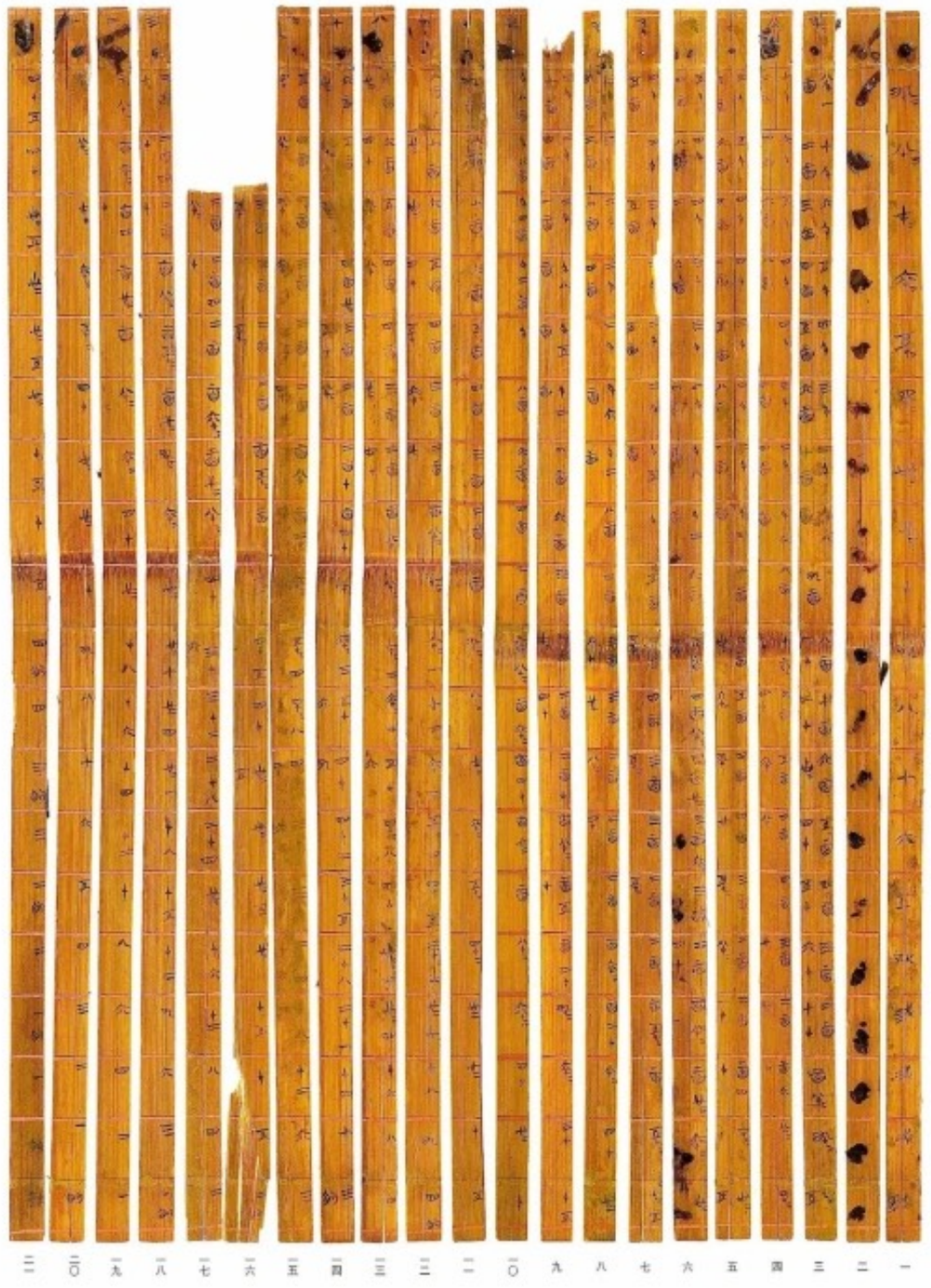
In other cultures, understanding was incomplete.

Video: Bees

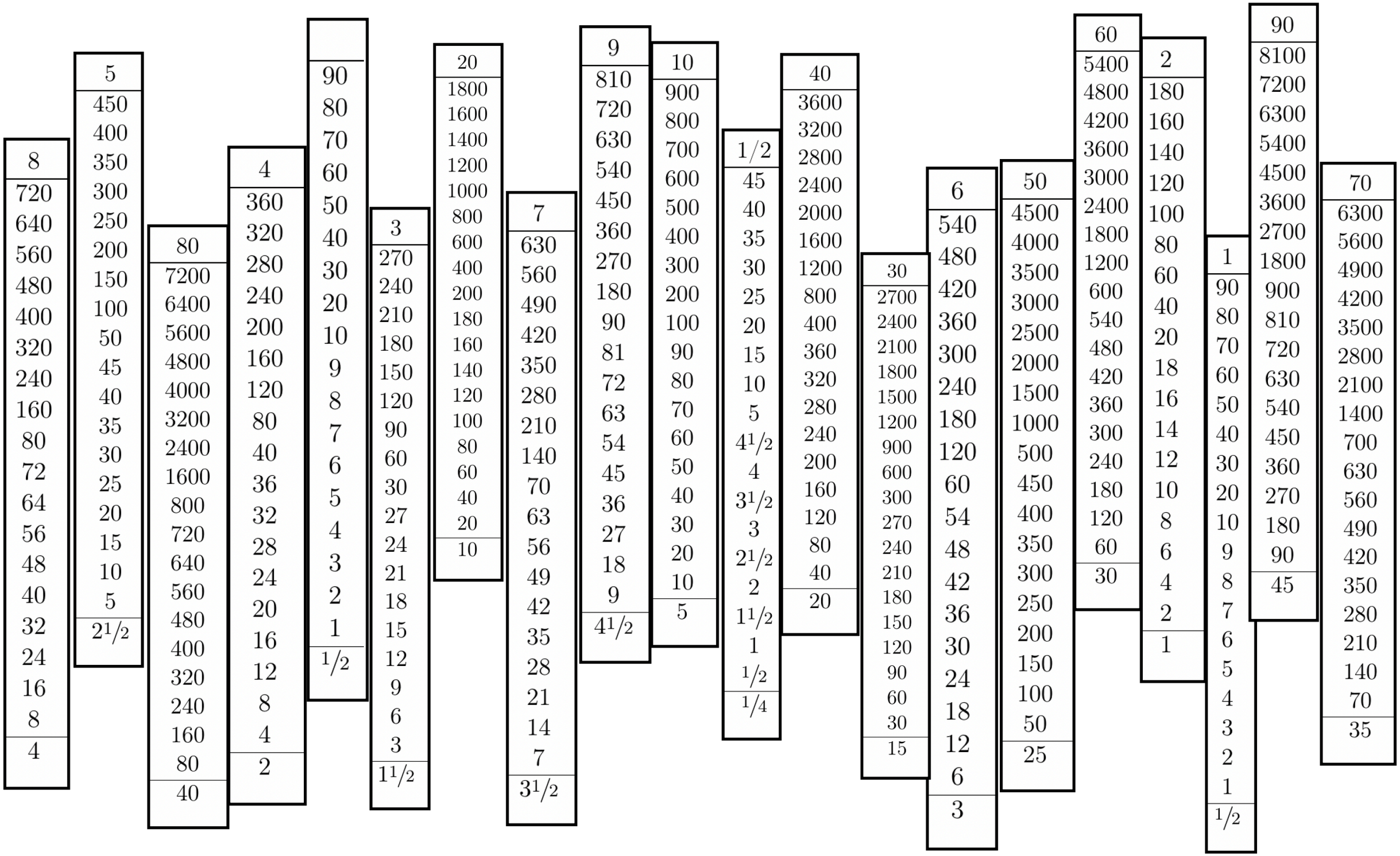
Video: Bees

Think Like A Math Historian

Think Like A Math Historian



Think Like A Math Historian



Think Like A Math Historian

1/2	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	
45	90	180	270	360	450	540	630	720	810	900	1800	2700	3600	4500	5400	6300	7200	8100	90
40	80	160	240	320	400	480	560	640	720	800	1600	2400	3200	4000	4800	5600	6400	7200	80
35	70	140	210	280	350	420	490	560	630	700	1400	2100	2800	3500	4200	4900	5600	6300	70
30	60	120	180	240	300	360	420	480	540	600	1200	1800	2400	3000	3600	4200	4800	5400	60
25	50	100	150	200	250	300	350	400	450	500	1000	1500	2000	2500	3000	3500	4000	4500	50
20	40	80	120	160	200	240	280	320	360	400	800	1200	1600	2000	2400	2800	3200	3600	40
15	30	60	90	120	150	180	210	240	270	300	600	900	1200	1500	1800	2100	2400	2700	30
10	20	40	60	80	100	120	140	160	180	200	400	600	800	1000	1200	1400	1600	1800	20
5	10	20	30	40	50	60	70	80	90	100	200	300	400	500	600	700	800	900	10
$4\frac{1}{2}$	9	18	27	36	45	54	63	72	81	90	180	270	360	450	540	630	720	810	9
4	8	16	24	32	40	48	56	64	72	80	160	240	320	400	480	560	640	720	8
$3\frac{1}{2}$	7	14	21	28	35	42	49	56	63	70	140	210	280	350	420	490	560	630	7
3	6	12	18	24	30	36	42	48	54	60	120	180	240	300	360	420	480	540	6
$2\frac{1}{2}$	5	10	15	20	25	30	35	40	45	50	100	150	200	250	300	350	400	450	5
2	4	8	12	16	20	24	28	32	36	40	80	120	160	200	240	280	320	360	4
$1\frac{1}{2}$	3	6	9	12	15	18	21	24	27	30	60	90	120	150	180	210	240	270	3
1	2	4	6	8	10	12	14	16	18	20	40	60	80	100	120	140	160	180	2
$\frac{1}{2}$	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	1
$\frac{1}{4}$	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$	5	10	15	20	25	30	35	40	45	$\frac{1}{2}$

Think Like A Math Historian

1/2	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	
45	90	180	270	360	450	540	630	720	810	900	1800	2700	3600	4500	5400	6300	7200	8100	90
40	80	160	240	320	400	480	560	640	720	800	1600	2400	3200	4000	4800	5600	6400	7200	80
35	70	140	210	280	350	420	490	560	630	700	1400	2100	2800	3500	4200	4900	5600	6300	70
30	60	120	180	240	300	360	420	480	540	600	1200	1800	2400	3000	3600	4200	4800	5400	60
25	50	100	150	200	250	300	350	400	450	500	1000	1500	2000	2500	3000	3500	4000	4500	50
20	40	80	120	160	200	240	280	320	360	400	800	1200	1600	2000	2400	2800	3200	3600	40
15	30	60	90	120	150	180	210	240	270	300	600	900	1200	1500	1800	2100	2400	2700	30
10	20	40	60	80	100	120	140	160	180	200	400	600	800	1000	1200	1400	1600	1800	20
5	10	20	30	40	50	60	70	80	90	100	200	300	400	500	600	700	800	900	10
$4\frac{1}{2}$	9	18	27	36	45	54	63	72	81	90	180	270	360	450	540	630	720	810	9
4	8	16	24	32	40	48	56	64	72	80	160	240	320	400	480	560	640	720	8
$3\frac{1}{2}$	7	14	21	28	35	42	49	56	63	70	140	210	280	350	420	490	560	630	7
3	6	12	18	24	30	36	42	48	54	60	120	180	240	300	360	420	480	540	6
$2\frac{1}{2}$	5	10	15	20	25	30	35	40	45	50	100	150	200	250	300	350	400	450	5
2	4	8	12	16	20	24	28	32	36	40	80	120	160	200	240	280	320	360	4
$1\frac{1}{2}$	3	6	9	12	15	18	21	24	27	30	60	90	120	150	180	210	240	270	3
1	2	4	6	8	10	12	14	16	18	20	40	60	80	100	120	140	160	180	2
$\frac{1}{2}$	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	1
$\frac{1}{4}$	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$	5	10	15	20	25	30	35	40	45	$\frac{1}{2}$

We will come back to this soon.

Arithmetic

Multiplication Tables

Multiplication Tables

- After counting comes arithmetic. Simple addition and subtraction could be done in head, on fingers, or with simple tools. Multiplication was probably much harder.

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MS 3866
Multiplication table for 1.12(=72).
Babylonia, 19th c. BC

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Multiplication Tables

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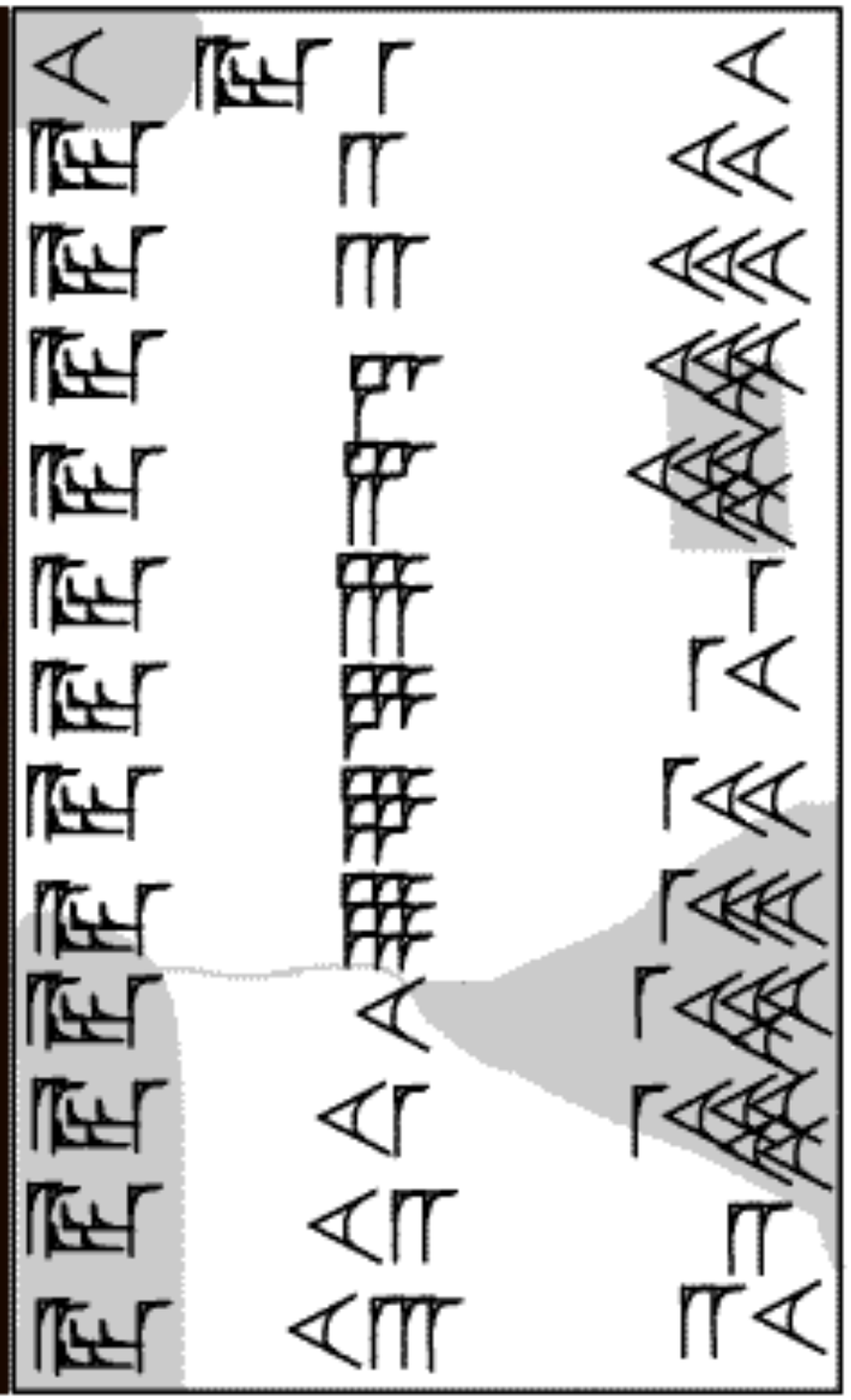

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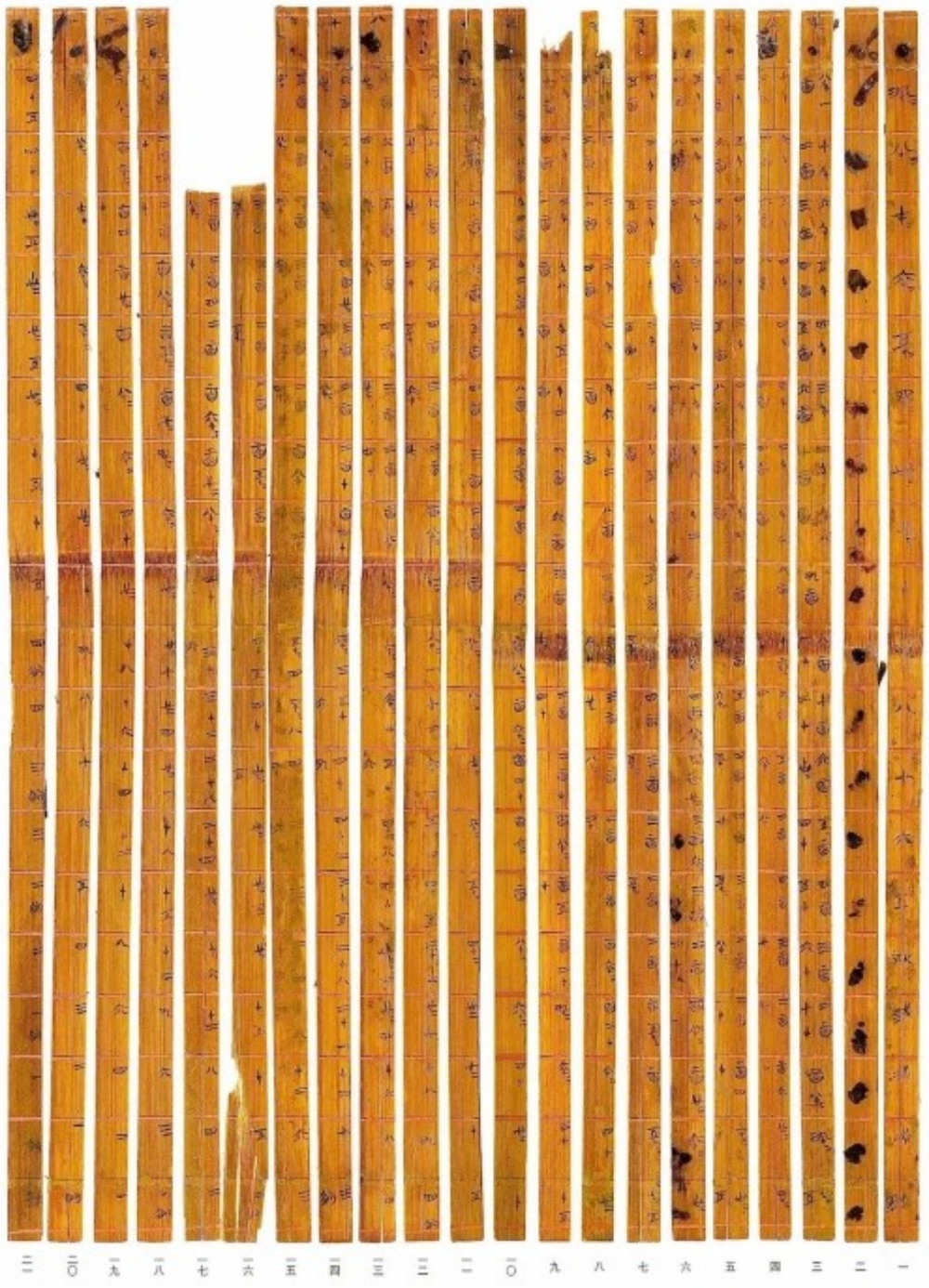
- First known *decimal* multiplication table is a 2,300 year old Chinese multiplication table.

- It was discovered in 2009!

Tsinghua Bamboo Strips



Tsinghua Bamboo Strips

- An ancient multiplication table, written on bamboo strips and found in China, is the first known base-10 multiplication table.
- With a little extra work, one can use it to compute the product of any two integers or half integers from 0.5 to 99.5.
  That is, you can multiply together any two numbers from this list: 0.5, 1, 1.5, 2, 2.5, … , 99.5.

Tsinghua Bamboo Strips

- This multiplication table can compute the product of any two numbers from the set {0.5,1,1.5,2,2.5,3,3.5,…,99.5}. How?

1/2	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	
45	90	180	270	360	450	540	630	720	810	900	1800	2700	3600	4500	5400	6300	7200	8100	90
40	80	160	240	320	400	480	560	640	720	800	1600	2400	3200	4000	4800	5600	6400	7200	80
35	70	140	210	280	350	420	490	560	630	700	1400	2100	2800	3500	4200	4900	5600	6300	70
30	60	120	180	240	300	360	420	480	540	600	1200	1800	2400	3000	3600	4200	4800	5400	60
25	50	100	150	200	250	300	350	400	450	500	1000	1500	2000	2500	3000	3500	4000	4500	50
20	40	80	120	160	200	240	280	320	360	400	800	1200	1600	2000	2400	2800	3200	3600	40
15	30	60	90	120	150	180	210	240	270	300	600	900	1200	1500	1800	2100	2400	2700	30
10	20	40	60	80	100	120	140	160	180	200	400	600	800	1000	1200	1400	1600	1800	20
5	10	20	30	40	50	60	70	80	90	100	200	300	400	500	600	700	800	900	10
4 1/2	9	18	27	36	45	54	63	72	81	90	180	270	360	450	540	630	720	810	9
4	8	16	24	32	40	48	56	64	72	80	160	240	320	400	480	560	640	720	8
3 1/2	7	14	21	28	35	42	49	56	63	70	140	210	280	350	420	490	560	630	7
3	6	12	18	24	30	36	42	48	54	60	120	180	240	300	360	420	480	540	6
2 1/2	5	10	15	20	25	30	35	40	45	50	100	150	200	250	300	350	400	450	5
2	4	8	12	16	20	24	28	32	36	40	80	120	160	200	240	280	320	360	4
1 1/2	3	6	9	12	15	18	21	24	27	30	60	90	120	150	180	210	240	270	3
1	2	4	6	8	10	12	14	16	18	20	40	60	80	100	120	140	160	180	2
1/2	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	1
1/4	1/2	1	1 1/2	2	2 1/2	3	3 1/2	4	4 1/2	5	10	15	20	25	30	35	40	45	1/2

Think Like A Math Historian

Tsinghua Bamboo Strips

- This multiplication table can compute the product of any two numbers from the set {0.5,1,1.5,2,2.5,3,3.5,…,99.5}. How?

1/2	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	
45	90	180	270	360	450	540	630	720	810	900	1800	2700	3600	4500	5400	6300	7200	8100	90
40	80	160	240	320	400	480	560	640	720	800	1600	2400	3200	4000	4800	5600	6400	7200	80
35	70	140	210	280	350	420	490	560	630	700	1400	2100	2800	3500	4200	4900	5600	6300	70
30	60	120	180	240	300	360	420	480	540	600	1200	1800	2400	3000	3600	4200	4800	5400	60
25	50	100	150	200	250	300	350	400	450	500	1000	1500	2000	2500	3000	3500	4000	4500	50
20	40	80	120	160	200	240	280	320	360	400	800	1200	1600	2000	2400	2800	3200	3600	40
15	30	60	90	120	150	180	210	240	270	300	600	900	1200	1500	1800	2100	2400	2700	30
10	20	40	60	80	100	120	140	160	180	200	400	600	800	1000	1200	1400	1600	1800	20
5	10	20	30	40	50	60	70	80	90	100	200	300	400	500	600	700	800	900	10
4 1/2	9	18	27	36	45	54	63	72	81	90	180	270	360	450	540	630	720	810	9
4	8	16	24	32	40	48	56	64	72	80	160	240	320	400	480	560	640	720	8
3 1/2	7	14	21	28	35	42	49	56	63	70	140	210	280	350	420	490	560	630	7
3	6	12	18	24	30	36	42	48	54	60	120	180	240	300	360	420	480	540	6
2 1/2	5	10	15	20	25	30	35	40	45	50	100	150	200	250	300	350	400	450	5
2	4	8	12	16	20	24	28	32	36	40	80	120	160	200	240	280	320	360	4
1 1/2	3	6	9	12	15	18	21	24	27	30	60	90	120	150	180	210	240	270	3
1	2	4	6	8	10	12	14	16	18	20	40	60	80	100	120	140	160	180	2
1/2	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	1
1/4	1/2	1	1 1/2	2	2 1/2	3	3 1/2	4	4 1/2	5	10	15	20	25	30	35	40	45	1/2

Tsinghua Bamboo Strips

- Example: $24 \cdot 36.5 = ?$

1/2	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	
45	90	180	270	360	450	540	630	720	810	900	1800	2700	3600	4500	5400	6300	7200	8100	90
40	80	160	240	320	400	480	560	640	720	800	1600	2400	3200	4000	4800	5600	6400	7200	80
35	70	140	210	280	350	420	490	560	630	700	1400	2100	2800	3500	4200	4900	5600	6300	70
30	60	120	180	240	300	360	420	480	540	600	1200	1800	2400	3000	3600	4200	4800	5400	60
25	50	100	150	200	250	300	350	400	450	500	1000	1500	2000	2500	3000	3500	4000	4500	50
20	40	80	120	160	200	240	280	320	360	400	800	1200	1600	2000	2400	2800	3200	3600	40
15	30	60	90	120	150	180	210	240	270	300	600	900	1200	1500	1800	2100	2400	2700	30
10	20	40	60	80	100	120	140	160	180	200	400	600	800	1000	1200	1400	1600	1800	20
5	10	20	30	40	50	60	70	80	90	100	200	300	400	500	600	700	800	900	10
4 1/2	9	18	27	36	45	54	63	72	81	90	180	270	360	450	540	630	720	810	9
4	8	16	24	32	40	48	56	64	72	80	160	240	320	400	480	560	640	720	8
3 1/2	7	14	21	28	35	42	49	56	63	70	140	210	280	350	420	490	560	630	7
3	6	12	18	24	30	36	42	48	54	60	120	180	240	300	360	420	480	540	6
2 1/2	5	10	15	20	25	30	35	40	45	50	100	150	200	250	300	350	400	450	5
2	4	8	12	16	20	24	28	32	36	40	80	120	160	200	240	280	320	360	4
1 1/2	3	6	9	12	15	18	21	24	27	30	60	90	120	150	180	210	240	270	3
1	2	4	6	8	10	12	14	16	18	20	40	60	80	100	120	140	160	180	2
1/2	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	1
1/4	1/2	1	1 1/2	2	2 1/2	3	3 1/2	4	4 1/2	5	10	15	20	25	30	35	40	45	1/2

Tsinghua Bamboo Strips

- Example: $24 \cdot 36.5 = ?$

1/2	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	
45	90	180	270	360	450	540	630	720	810	900	1800	2700	3600	4500	5400	6300	7200	8100	90
40	80	160	240	320	400	480	560	640	720	800	1600	2400	3200	4000	4800	5600	6400	7200	80
35	70	140	210	280	350	420	490	560	630	700	1400	2100	2800	3500	4200	4900	5600	6300	70
30	60	120	180	240	300	360	420	480	540	600	1200	1800	2400	3000	3600	4200	4800	5400	60
25	50	100	150	200	250	300	350	400	450	500	1000	1500	2000	2500	3000	3500	4000	4500	50
20	40	80	120	160	200	240	280	320	360	400	800	1200	1600	2000	2400	2800	3200	3600	40
15	30	60	90	120	150	180	210	240	270	300	600	900	1200	1500	1800	2100	2400	2700	30
10	20	40	60	80	100	120	140	160	180	200	400	600	800	1000	1200	1400	1600	1800	20
5	10	20	30	40	50	60	70	80	90	100	200	300	400	500	600	700	800	900	10
4 1/2	9	18	27	36	45	54	63	72	81	90	180	270	360	450	540	630	720	810	9
4	8	16	24	32	40	48	56	64	72	80	160	240	320	400	480	560	640	720	8
3 1/2	7	14	21	28	35	42	49	56	63	70	140	210	280	350	420	490	560	630	7
3	6	12	18	24	30	36	42	48	54	60	120	180	240	300	360	420	480	540	6
2 1/2	5	10	15	20	25	30	35	40	45	50	100	150	200	250	300	350	400	450	5
2	4	8	12	16	20	24	28	32	36	40	80	120	160	200	240	280	320	360	4
1 1/2	3	6	9	12	15	18	21	24	27	30	60	90	120	150	180	210	240	270	3
1	2	4	6	8	10	12	14	16	18	20	40	60	80	100	120	140	160	180	2
1/2	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	1
1/4	1/2	1	1 1/2	2	2 1/2	3	3 1/2	4	4 1/2	5	10	15	20	25	30	35	40	45	1/2

Tsinghua Bamboo Strips

- Example: $24 \cdot 36.5 = ?$

1/2	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	
45	90	180	270	360	450	540	630	720	810	900	1800	2700	3600	4500	5400	6300	7200	8100	90
40	80	160	240	320	400	480	560	640	720	800	1600	2400	3200	4000	4800	5600	6400	7200	80
35	70	140	210	280	350	420	490	560	630	700	1400	2100	2800	3500	4200	4900	5600	6300	70
30	60	120	180	240	300	360	420	480	540	600	1200	1800	2400	3000	3600	4200	4800	5400	60
25	50	100	150	200	250	300	350	400	450	500	1000	1500	2000	2500	3000	3500	4000	4500	50
20	40	80	120	160	200	240	280	320	360	400	800	1200	1600	2000	2400	2800	3200	3600	40
15	30	60	90	120	150	180	210	240	270	300	600	900	1200	1500	1800	2100	2400	2700	30
10	20	40	60	80	100	120	140	160	180	200	400	600	800	1000	1200	1400	1600	1800	20
5	10	20	30	40	50	60	70	80	90	100	200	300	400	500	600	700	800	900	10
4 1/2	9	18	27	36	45	54	63	72	81	90	180	270	360	450	540	630	720	810	9
4	8	16	24	32	40	48	56	64	72	80	160	240	320	400	480	560	640	720	8
3 1/2	7	14	21	28	35	42	49	56	63	70	140	210	280	350	420	490	560	630	7
3	6	12	18	24	30	36	42	48	54	60	120	180	240	300	360	420	480	540	6
2 1/2	5	10	15	20	25	30	35	40	45	50	100	150	200	250	300	350	400	450	5
2	4	8	12	16	20	24	28	32	36	40	80	120	160	200	240	280	320	360	4
1 1/2	3	6	9	12	15	18	21	24	27	30	60	90	120	150	180	210	240	270	3
1	2	4	6	8	10	12	14	16	18	20	40	60	80	100	120	140	160	180	2
1/2	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	1
1/4	1/2	1	1 1/2	2	2 1/2	3	3 1/2	4	4 1/2	5	10	15	20	25	30	35	40	45	1/2

Tsinghua Bamboo Strips

- Example: $24 \cdot 36.5 = ?$

1/2	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	
45	90	180	270	360	450	540	630	720	810	900	1800	2700	3600	4500	5400	6300	7200	8100	90
40	80	160	240	320	400	480	560	640	720	800	1600	2400	3200	4000	4800	5600	6400	7200	80
35	70	140	210	280	350	420	490	560	630	700	1400	2100	2800	3500	4200	4900	5600	6300	70
30	60	120	180	240	300	360	420	480	540	600	1200	1800	2400	3000	3600	4200	4800	5400	60
25	50	100	150	200	250	300	350	400	450	500	1000	1500	2000	2500	3000	3500	4000	4500	50
20	40	80	120	160	200	240	280	320	360	400	800	1200	1600	2000	2400	2800	3200	3600	40
15	30	60	90	120	150	180	210	240	270	300	600	900	1200	1500	1800	2100	2400	2700	30
10	20	40	60	80	100	120	140	160	180	200	400	600	800	1000	1200	1400	1600	1800	20
5	10	20	30	40	50	60	70	80	90	100	200	300	400	500	600	700	800	900	10
$4\frac{1}{2}$	9	18	27	36	45	54	63	72	81	90	180	270	360	450	540	630	720	810	9
4	8	16	24	32	40	48	56	64	72	80	160	240	320	400	480	560	640	720	8
$3\frac{1}{2}$	7	14	21	28	35	42	49	56	63	70	140	210	280	350	420	490	560	630	7
3	6	12	18	24	30	36	42	48	54	60	120	180	240	300	360	420	480	540	6
$2\frac{1}{2}$	5	10	15	20	25	30	35	40	45	50	100	150	200	250	300	350	400	450	5
2	4	8	12	16	20	24	28	32	36	40	80	120	160	200	240	280	320	360	4
$1\frac{1}{2}$	3	6	9	12	15	18	21	24	27	30	60	90	120	150	180	210	240	270	3
1	2	4	6	8	10	12	14	16	18	20	40	60	80	100	120	140	160	180	2
$\frac{1}{2}$	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	1
$\frac{1}{4}$	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$	5	10	15	20	25	30	35	40	45	$\frac{1}{2}$

Tsinghua Bamboo Strips

- Example: $24 \cdot 36.5 = ?$

1/2	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	
45	90	180	270	360	450	540	630	720	810	900	1800	2700	3600	4500	5400	6300	7200	8100	90
40	80	160	240	320	400	480	560	640	720	800	1600	2400	3200	4000	4800	5600	6400	7200	80
35	70	140	210	280	350	420	490	560	630	700	1400	2100	2800	3500	4200	4900	5600	6300	70
30	60	120	180	240	300	360	420	480	540	600	1200	1800	2400	3000	3600	4200	4800	5400	60
25	50	100	150	200	250	300	350	400	450	500	1000	1500	2000	2500	3000	3500	4000	4500	50
20	40	80	120	160	200	240	280	320	360	400	800	1200	1600	2000	2400	2800	3200	3600	40
15	30	60	90	120	150	180	210	240	270	300	600	900	1200	1500	1800	2100	2400	2700	30
10	20	40	60	80	100	120	140	160	180	200	400	600	800	1000	1200	1400	1600	1800	20
5	10	20	30	40	50	60	70	80	90	100	200	300	400	500	600	700	800	900	10
$4\frac{1}{2}$	9	18	27	36	45	54	63	72	81	90	180	270	360	450	540	630	720	810	9
4	8	16	24	32	40	48	56	64	72	80	160	240	320	400	480	560	640	720	8
$3\frac{1}{2}$	7	14	21	28	35	42	49	56	63	70	140	210	280	350	420	490	560	630	7
3	6	12	18	24	30	36	42	48	54	60	120	180	240	300	360	420	480	540	6
$2\frac{1}{2}$	5	10	15	20	25	30	35	40	45	50	100	150	200	250	300	350	400	450	5
2	4	8	12	16	20	24	28	32	36	40	80	120	160	200	240	280	320	360	4
$1\frac{1}{2}$	3	6	9	12	15	18	21	24	27	30	60	90	120	150	180	210	240	270	3
1	2	4	6	8	10	12	14	16	18	20	40	60	80	100	120	140	160	180	2
$\frac{1}{2}$	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	1
$\frac{1}{4}$	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$	5	10	15	20	25	30	35	40	45	$\frac{1}{2}$

Tsinghua Bamboo Strips

- Example: $24 \cdot 36.5 = ?$

1/2	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	
45	90	180	270	360	450	540	630	720	810	900	1800	2700	3600	4500	5400	6300	7200	8100	90
40	80	160	240	320	400	480	560	640	720	800	1600	2400	3200	4000	4800	5600	6400	7200	80
35	70	140	210	280	350	420	490	560	630	700	1400	2100	2800	3500	4200	4900	5600	6300	70
30	60	120	180	240	300	360	420	480	540	600	1200	1800	2400	3000	3600	4200	4800	5400	60
25	50	100	150	200	250	300	350	400	450	500	1000	1500	2000	2500	3000	3500	4000	4500	50
20	40	80	120	160	200	240	280	320	360	400	800	1200	1600	2000	2400	2800	3200	3600	40
15	30	60	90	120	150	180	210	240	270	300	600	900	1200	1500	1800	2100	2400	2700	30
10	20	40	60	80	100	120	140	160	180	200	400	600	800	1000	1200	1400	1600	1800	20
5	10	20	30	40	50	60	70	80	90	100	200	300	400	500	600	700	800	900	10
$4\frac{1}{2}$	9	18	27	36	45	54	63	72	81	90	180	270	360	450	540	630	720	810	9
4	8	16	24	32	40	48	56	64	72	80	160	240	320	400	480	560	640	720	8
$3\frac{1}{2}$	7	14	21	28	35	42	49	56	63	70	140	210	280	350	420	490	560	630	7
3	6	12	18	24	30	36	42	48	54	60	120	180	240	300	360	420	480	540	6
$2\frac{1}{2}$	5	10	15	20	25	30	35	40	45	50	100	150	200	250	300	350	400	450	5
2	4	8	12	16	20	24	28	32	36	40	80	120	160	200	240	280	320	360	4
$1\frac{1}{2}$	3	6	9	12	15	18	21	24	27	30	60	90	120	150	180	210	240	270	3
1	2	4	6	8	10	12	14	16	18	20	40	60	80	100	120	140	160	180	2
$\frac{1}{2}$	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	1
$\frac{1}{4}$	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$	5	10	15	20	25	30	35	40	45	$\frac{1}{2}$

Tsinghua Bamboo Strips

- Example: $24 \cdot 36.5 = ?$

1/2	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	
45	90	180	270	360	450	540	630	720	810	900	1800	2700	3600	4500	5400	6300	7200	8100	90
40	80	160	240	320	400	480	560	640	720	800	1600	2400	3200	4000	4800	5600	6400	7200	80
35	70	140	210	280	350	420	490	560	630	700	1400	2100	2800	3500	4200	4900	5600	6300	70
30	60	120	180	240	300	360	420	480	540	600	1200	1800	2400	3000	3600	4200	4800	5400	60
25	50	100	150	200	250	300	350	400	450	500	1000	1500	2000	2500	3000	3500	4000	4500	50
20	40	80	120	160	200	240	280	320	360	400	800	1200	1600	2000	2400	2800	3200	3600	40
15	30	60	90	120	150	180	210	240	270	300	600	900	1200	1500	1800	2100	2400	2700	30
10	20	40	60	80	100	120	140	160	180	200	400	600	800	1000	1200	1400	1600	1800	20
5	10	20	30	40	50	60	70	80	90	100	200	300	400	500	600	700	800	900	10
$4\frac{1}{2}$	9	18	27	36	45	54	63	72	81	90	180	270	360	450	540	630	720	810	9
4	8	16	24	32	40	48	56	64	72	80	160	240	320	400	480	560	640	720	8
$3\frac{1}{2}$	7	14	21	28	35	42	49	56	63	70	140	210	280	350	420	490	560	630	7
3	6	12	18	24	30	36	42	48	54	60	120	180	240	300	360	420	480	540	6
$2\frac{1}{2}$	5	10	15	20	25	30	35	40	45	50	100	150	200	250	300	350	400	450	5
2	4	8	12	16	20	24	28	32	36	40	80	120	160	200	240	280	320	360	4
$1\frac{1}{2}$	3	6	9	12	15	18	21	24	27	30	60	90	120	150	180	210	240	270	3
1	2	4	6	8	10	12	14	16	18	20	40	60	80	100	120	140	160	180	2
$\frac{1}{2}$	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	1
$\frac{1}{4}$	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$	5	10	15	20	25	30	35	40	45	$\frac{1}{2}$

$$24 \cdot 36.5 = (20 + 4) \cdot (30 + 6 + 0.5)$$

Tsinghua Bamboo Strips

- Example: $24 \cdot 36.5 = ?$

1/2	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	
45	90	180	270	360	450	540	630	720	810	900	1800	2700	3600	4500	5400	6300	7200	8100	90
40	80	160	240	320	400	480	560	640	720	800	1600	2400	3200	4000	4800	5600	6400	7200	80
35	70	140	210	280	350	420	490	560	630	700	1400	2100	2800	3500	4200	4900	5600	6300	70
30	60	120	180	240	300	360	420	480	540	600	1200	1800	2400	3000	3600	4200	4800	5400	60
25	50	100	150	200	250	300	350	400	450	500	1000	1500	2000	2500	3000	3500	4000	4500	50
20	40	80	120	160	200	240	280	320	360	400	800	1200	1600	2000	2400	2800	3200	3600	40
15	30	60	90	120	150	180	210	240	270	300	600	900	1200	1500	1800	2100	2400	2700	30
10	20	40	60	80	100	120	140	160	180	200	400	600	800	1000	1200	1400	1600	1800	20
5	10	20	30	40	50	60	70	80	90	100	200	300	400	500	600	700	800	900	10
$4\frac{1}{2}$	9	18	27	36	45	54	63	72	81	90	180	270	360	450	540	630	720	810	9
4	8	16	24	32	40	48	56	64	72	80	160	240	320	400	480	560	640	720	8
$3\frac{1}{2}$	7	14	21	28	35	42	49	56	63	70	140	210	280	350	420	490	560	630	7
3	6	12	18	24	30	36	42	48	54	60	120	180	240	300	360	420	480	540	6
$2\frac{1}{2}$	5	10	15	20	25	30	35	40	45	50	100	150	200	250	300	350	400	450	5
2	4	8	12	16	20	24	28	32	36	40	80	120	160	200	240	280	320	360	4
$1\frac{1}{2}$	3	6	9	12	15	18	21	24	27	30	60	90	120	150	180	210	240	270	3
1	2	4	6	8	10	12	14	16	18	20	40	60	80	100	120	140	160	180	2
1/2	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	1
1/4	1/2	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$	5	10	15	20	25	30	35	40	45	1/2

$$24 \cdot 36.5 = (20 + 4) \cdot (30 + 6 + 0.5)$$

Tsinghua Bamboo Strips

- Example: $24 \cdot 36.5 = ?$

1/2	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	
45	90	180	270	360	450	540	630	720	810	900	1800	2700	3600	4500	5400	6300	7200	8100	90
40	80	160	240	320	400	480	560	640	720	800	1600	2400	3200	4000	4800	5600	6400	7200	80
35	70	140	210	280	350	420	490	560	630	700	1400	2100	2800	3500	4200	4900	5600	6300	70
30	60	120	180	240	300	360	420	480	540	600	1200	1800	2400	3000	3600	4200	4800	5400	60
25	50	100	150	200	250	300	350	400	450	500	1000	1500	2000	2500	3000	3500	4000	4500	50
20	40	80	120	160	200	240	280	320	360	400	800	1200	1600	2000	2400	2800	3200	3600	40
15	30	60	90	120	150	180	210	240	270	300	600	900	1200	1500	1800	2100	2400	2700	30
10	20	40	60	80	100	120	140	160	180	200	400	600	800	1000	1200	1400	1600	1800	20
5	10	20	30	40	50	60	70	80	90	100	200	300	400	500	600	700	800	900	10
$4\frac{1}{2}$	9	18	27	36	45	54	63	72	81	90	180	270	360	450	540	630	720	810	9
4	8	16	24	32	40	48	56	64	72	80	160	240	320	400	480	560	640	720	8
$3\frac{1}{2}$	7	14	21	28	35	42	49	56	63	70	140	210	280	350	420	490	560	630	7
3	6	12	18	24	30	36	42	48	54	60	120	180	240	300	360	420	480	540	6
$2\frac{1}{2}$	5	10	15	20	25	30	35	40	45	50	100	150	200	250	300	350	400	450	5
2	4	8	12	16	20	24	28	32	36	40	80	120	160	200	240	280	320	360	4
$1\frac{1}{2}$	3	6	9	12	15	18	21	24	27	30	60	90	120	150	180	210	240	270	3
1	2	4	6	8	10	12	14	16	18	20	40	60	80	100	120	140	160	180	2
$\frac{1}{2}$	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	1
$\frac{1}{4}$	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$	5	10	15	20	25	30	35	40	45	$\frac{1}{2}$

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Tsinghua Bamboo Strips

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45	90	180	270	360	450	540	630	720	810	900	1800	2700	3600	4500	5400	6300	7200	8100	90
40	80	160	240	320	400	480	560	640	720	800	1600	2400	3200	4000	4800	5600	6400	7200	80
35	70	140	210	280	350	420	490	560	630	700	1400	2100	2800	3500	4200	4900	5600	6300	70
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$2\frac{1}{2}$	5	10	15	20	25	30	35	40	45	50	100	150	200	250	300	350	400	450	5
2	4	8	12	16	20	24	28	32	36	40	80	120	160	200	240	280	320	360	4
$1\frac{1}{2}$	3	6	9	12	15	18	21	24	27	30	60	90	120	150	180	210	240	270	3
1	2	4	6	8	10	12	14	16	18	20	40	60	80	100	120	140	160	180	2
$\frac{1}{2}$	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	1
$\frac{1}{4}$	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$	5	10	15	20	25	30	35	40	45	$\frac{1}{2}$

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Tsinghua Bamboo Strips

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5	10	20	30	40	50	60	70	80	90	100	200	300	400	500	600	700	800	900	10
$4\frac{1}{2}$	9	18	27	36	45	54	63	72	81	90	180	270	360	450	540	630	720	810	9
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$1\frac{1}{2}$	3	6	9	12	15	18	21	24	27	30	60	90	120	150	180	210	240	270	3
1	2	4	6	8	10	12	14	16	18	20	40	60	80	100	120	140	160	180	2
$\frac{1}{2}$	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	1
$\frac{1}{4}$	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$	5	10	15	20	25	30	35	40	45	$\frac{1}{2}$

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Tsinghua Bamboo Strips

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$\frac{1}{2}$	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	1
$\frac{1}{4}$	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$	5	10	15	20	25	30	35	40	45	$\frac{1}{2}$

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Tsinghua Bamboo Strips

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1/2	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	
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10	20	40	60	80	100	120	140	160	180	200	400	600	800	1000	1200	1400	1600	1800	20
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$1\frac{1}{2}$	3	6	9	12	15	18	21	24	27	30	60	90	120	150	180	210	240	270	3
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$\frac{1}{2}$	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	1
$\frac{1}{4}$	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$	5	10	15	20	25	30	35	40	45	$\frac{1}{2}$

$$24 \cdot 36.5 = (20 + 4) \cdot (30 + 6 + 0.5)$$
$$= 20 \cdot 30 + 20 \cdot 6 + 20 \cdot 0.5 + 4 \cdot 30 + 4 \cdot 6 + 4 \cdot 0.5$$

Tsinghua Bamboo Strips

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1/2	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	
45	90	180	270	360	450	540	630	720	810	900	1800	2700	3600	4500	5400	6300	7200	8100	90
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30	60	120	180	240	300	360	420	480	540	600	1200	1800	2400	3000	3600	4200	4800	5400	60
25	50	100	150	200	250	300	350	400	450	500	1000	1500	2000	2500	3000	3500	4000	4500	50
20	40	80	120	160	200	240	280	320	360	400	800	1200	1600	2000	2400	2800	3200	3600	40
15	30	60	90	120	150	180	210	240	270	300	600	900	1200	1500	1800	2100	2400	2700	30
10	20	40	60	80	100	120	140	160	180	200	400	600	800	1000	1200	1400	1600	1800	20
5	10	20	30	40	50	60	70	80	90	100	200	300	400	500	600	700	800	900	10
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$1\frac{1}{2}$	3	6	9	12	15	18	21	24	27	30	60	90	120	150	180	210	240	270	3
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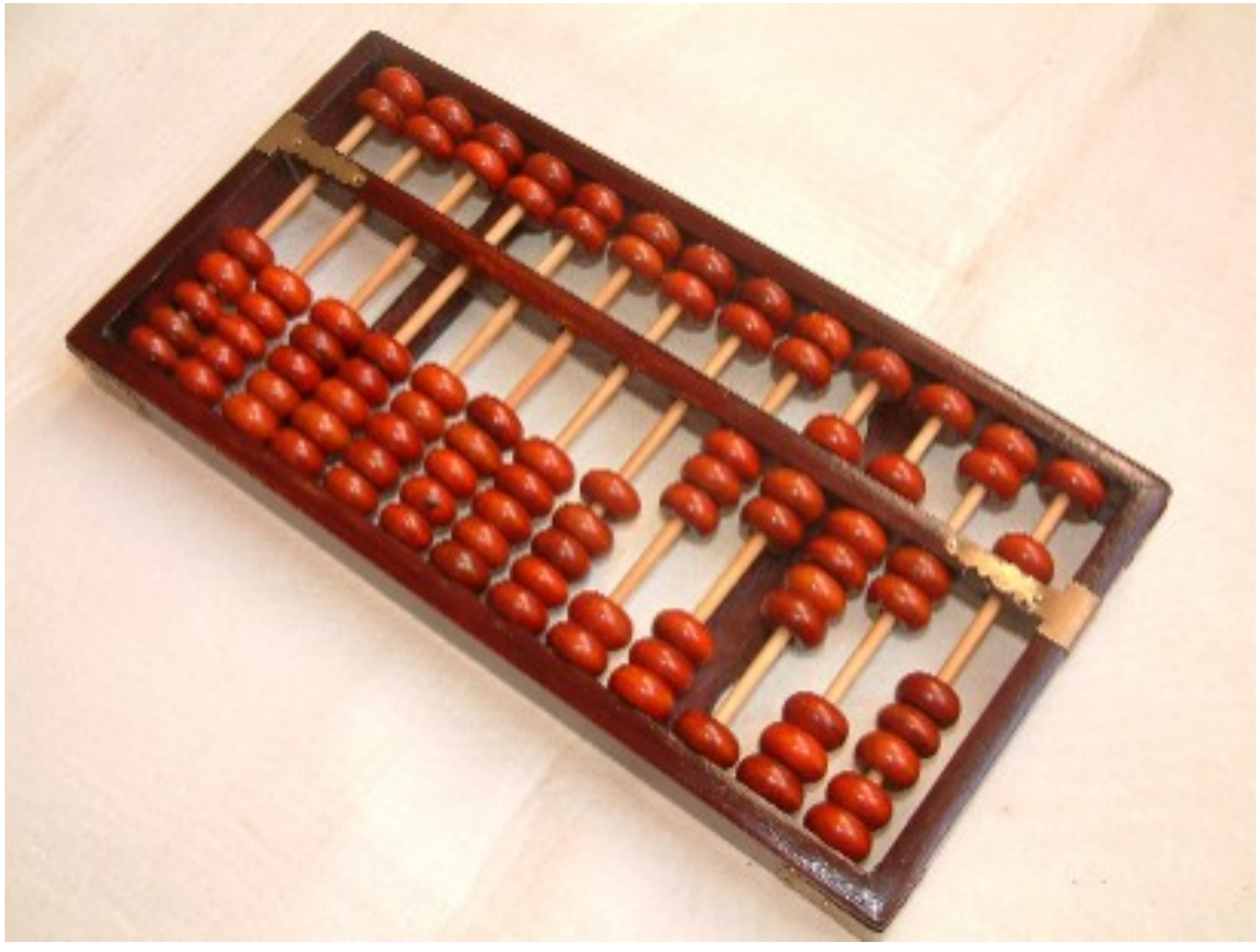
$\frac{1}{2}$	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	1
$\frac{1}{4}$	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$	5	10	15	20	25	30	35	40	45	$\frac{1}{2}$

$$24 \cdot 36.5 = (20 + 4) \cdot (30 + 6 + 0.5)$$
$$= 20 \cdot 30 + 20 \cdot 6 + 20 \cdot 0.5 + 4 \cdot 30 + 4 \cdot 6 + 4 \cdot 0.5$$
$$=$$ Look up on table, then add together

Ancient Calculators

Abacus



Counting Board

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- Appeared in many forms and many places.

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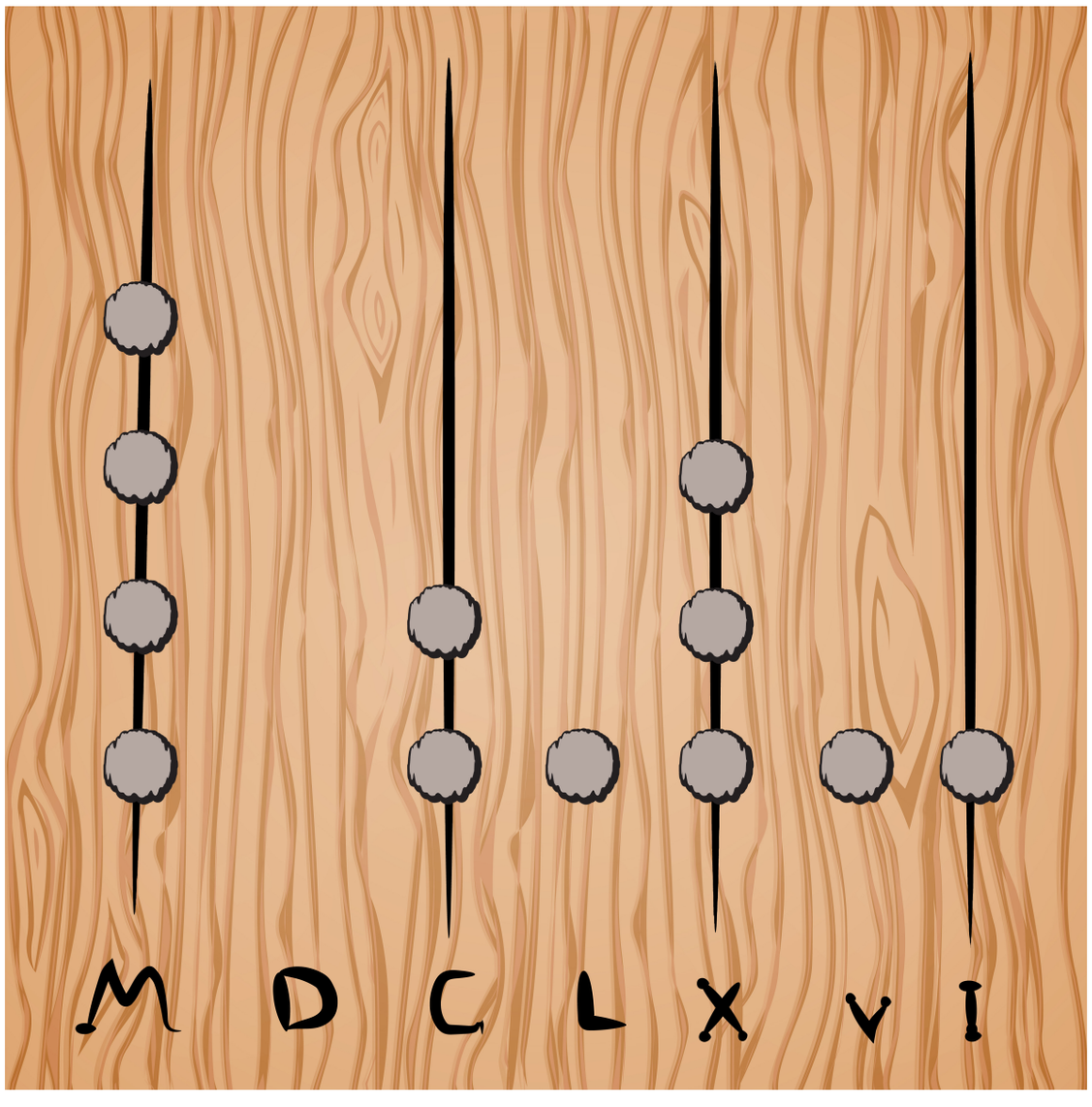
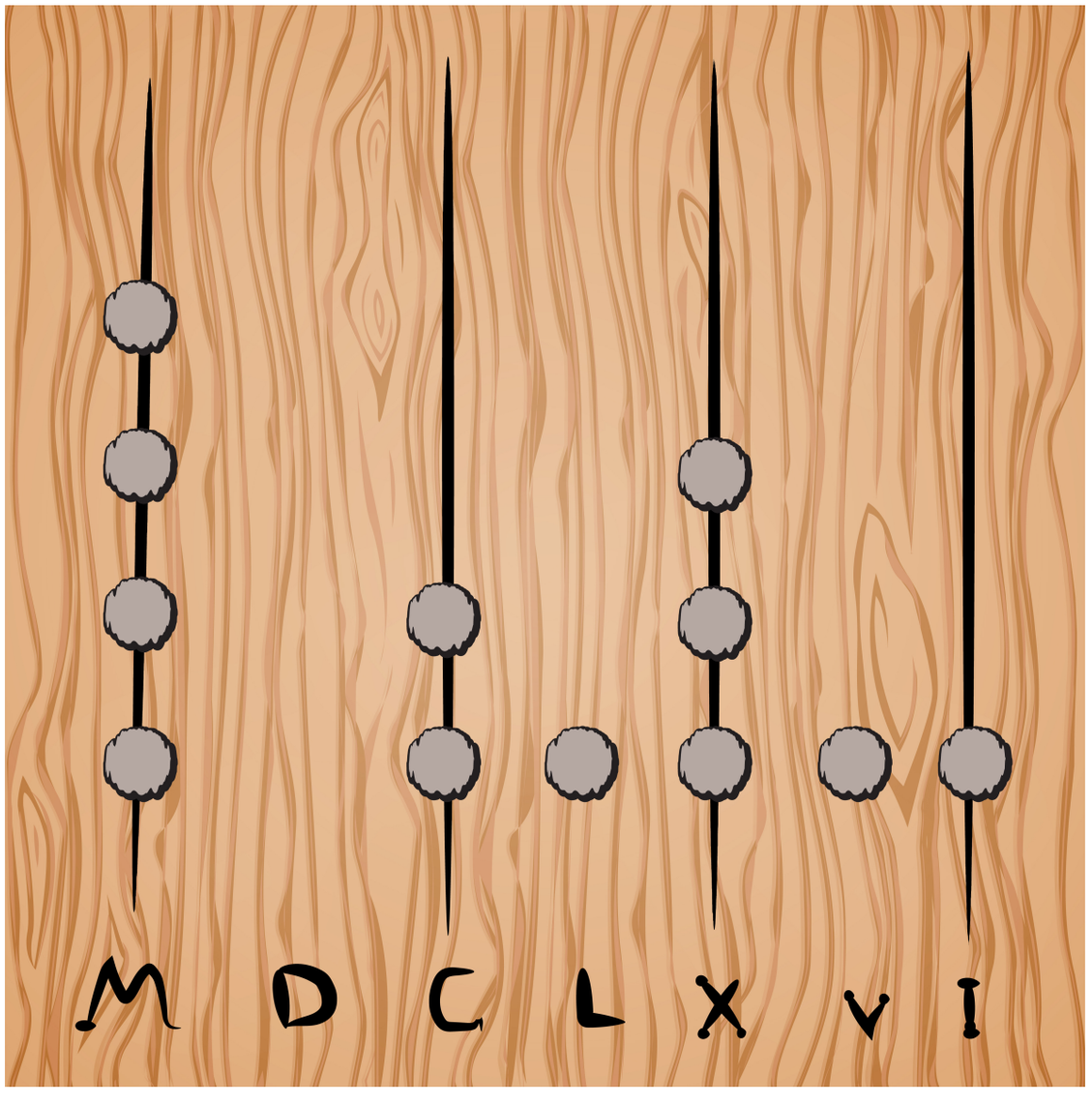
- This is what a Roman one might look like.



Represents 4,286

Counting Board

- Appeared in many forms and many places.
- This is what a Roman one might look like.
- Ancient Chinese mathematicians leveraged them to begin an important area of math. (Chapter 2)



Represents 4,286

The Incan Empire

Video: The Incan Empire

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- The Inca used what is called a *quipu*.

Video: Quipus

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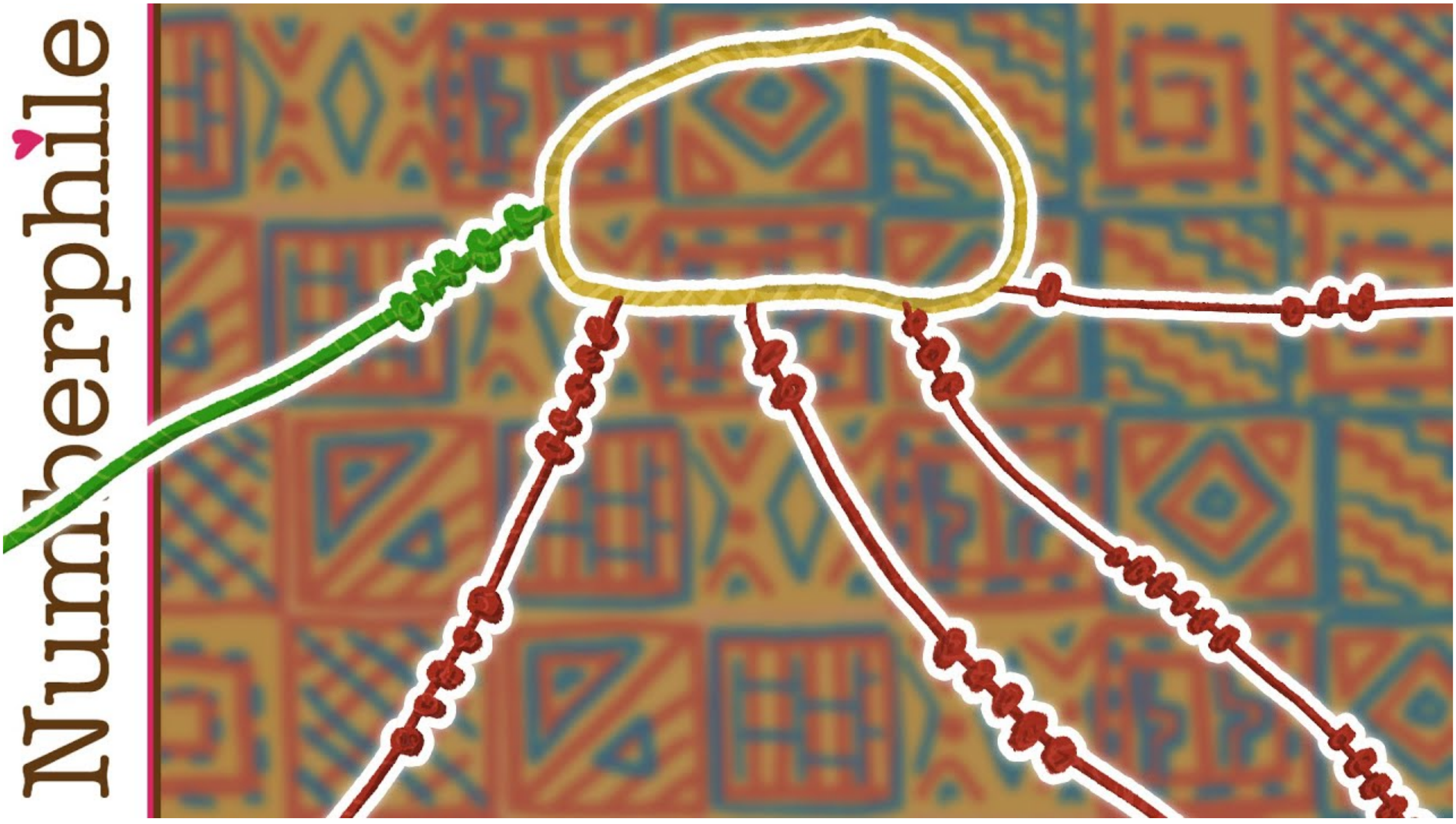
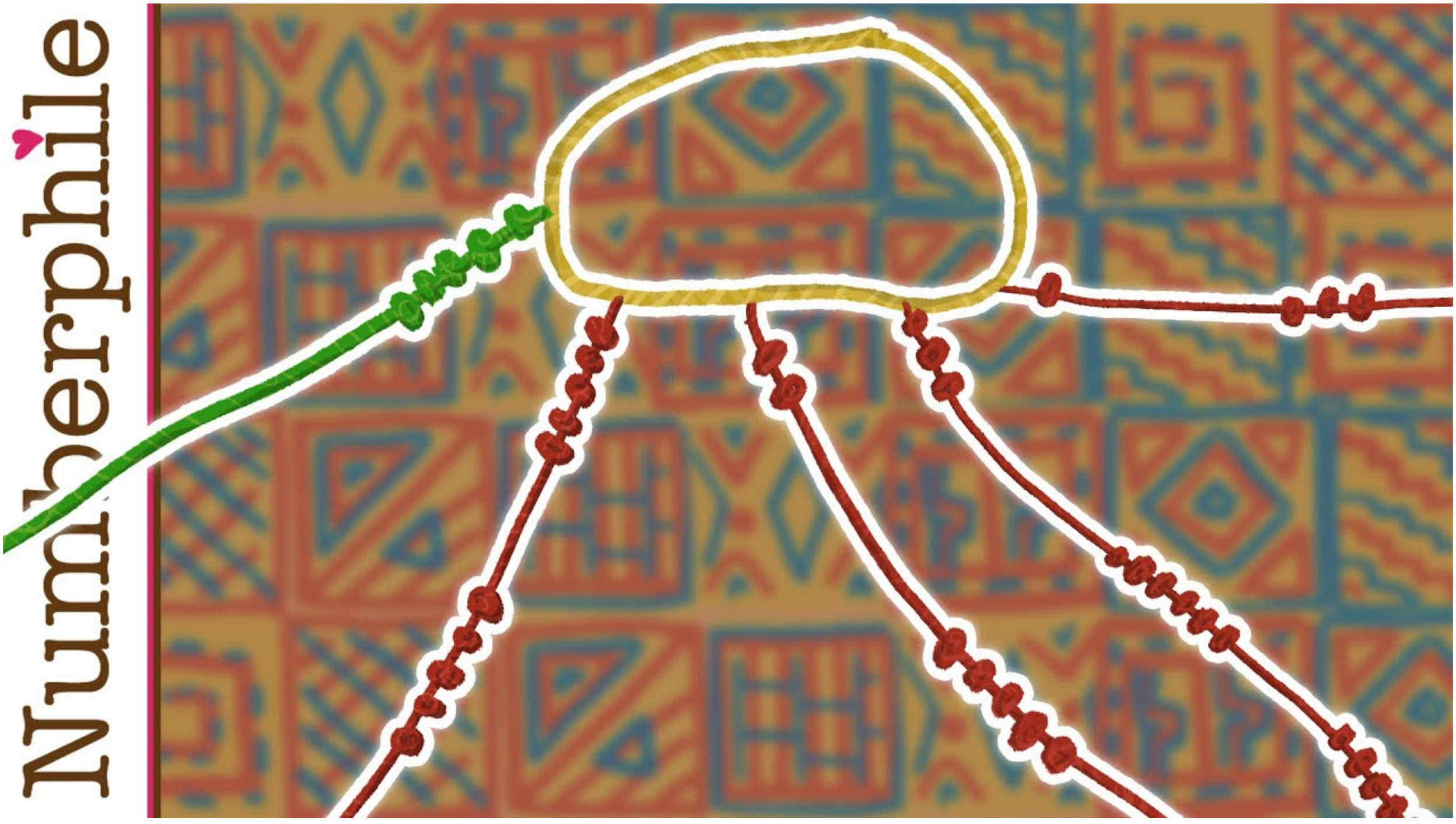


Video: Quipus



The Aftermath

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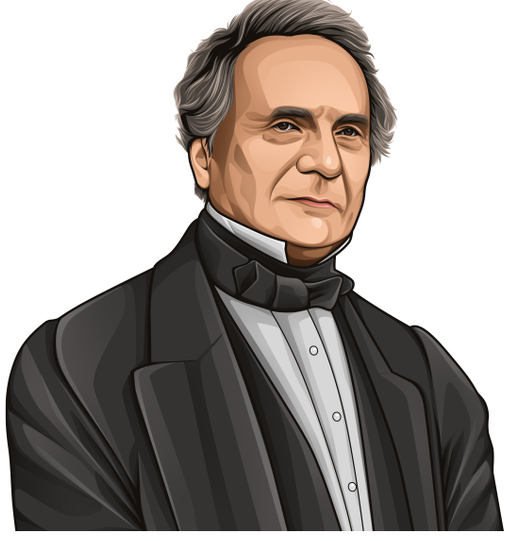
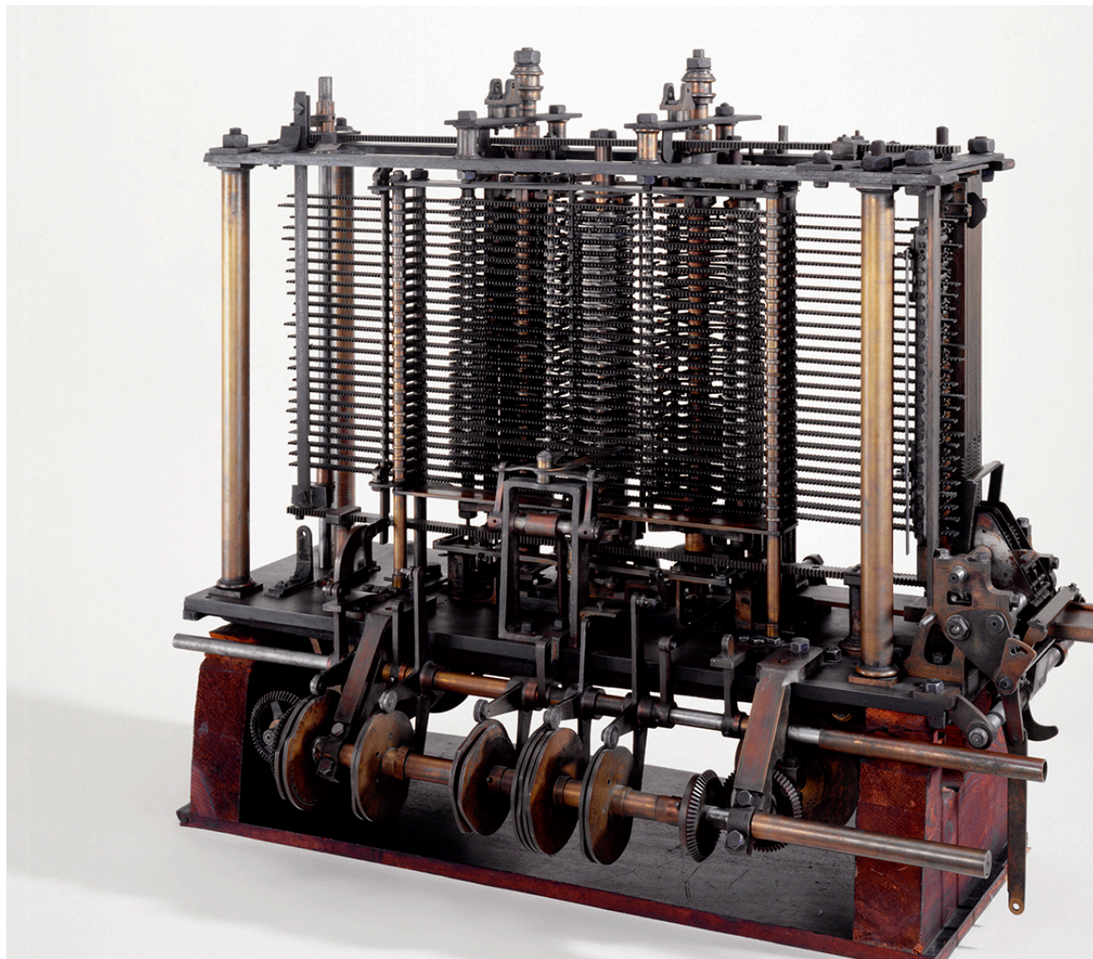


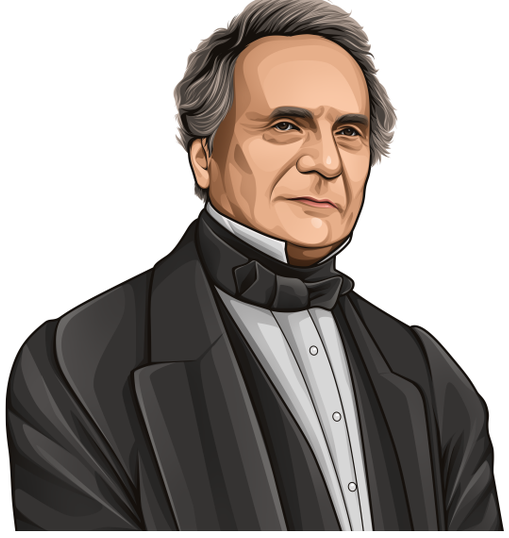
The Aftermath

- What about modern computers?

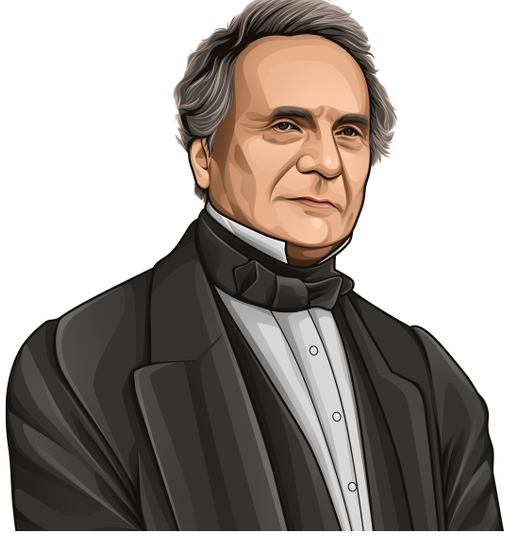
- Mechanical computers were developed before electric computers. They ran on punch cards.

- Charles Babbage invented the difference engine to perform computations. He later invented the analytical engine to do more complex tasks. It was the first “modern” computer. Portion of it pictured here —>



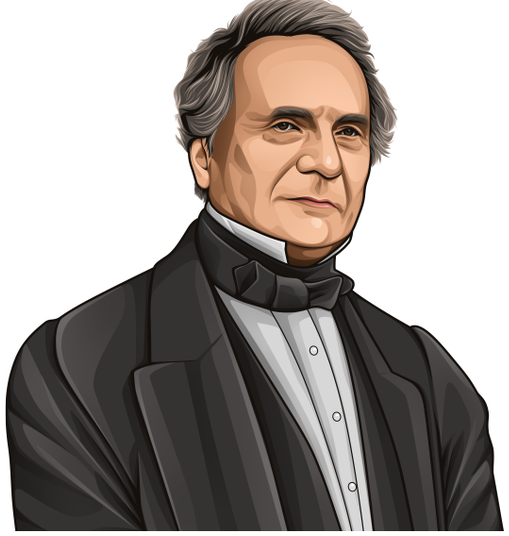


The Aftermath



The Aftermath

- Ada Lovelace realized its potential and became the world's first computer programmer.



The Aftermath

- Ada Lovelace realized its potential and became the world's first computer programmer.

- The first algorithm written specifically to be run on a computer was designed to compute Bernoulli numbers.

Diagram for the computation by the Engine of the Numbers of Bernoulli. See Note G. (page 722 *et seq.*)

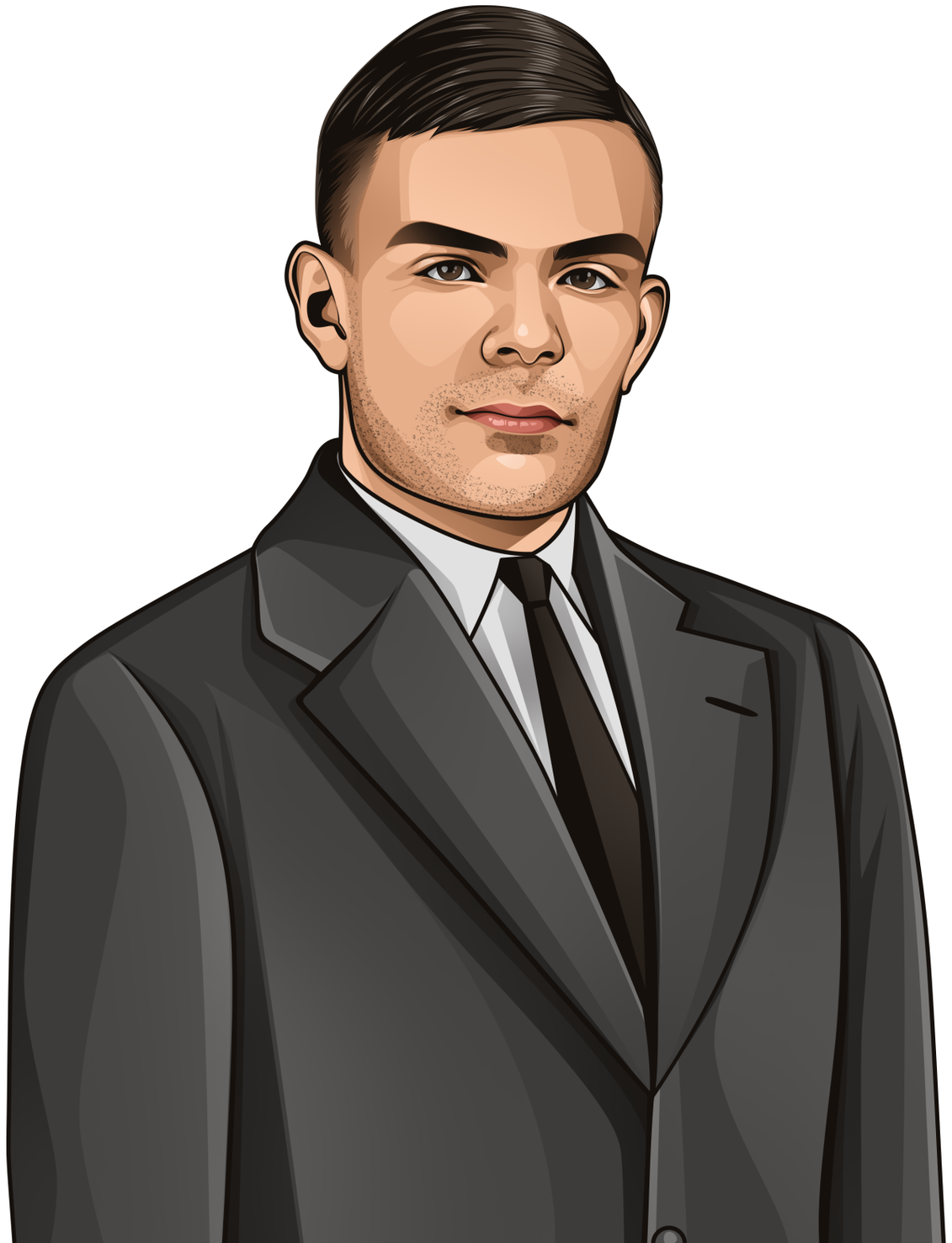
Number of Operation.	Nature of Operation.	Variables acted upon.	Variables receiving results.	Indication of change in the value on any Variable.	Statement of Results.	Data. ${}^1V_1$ ○ 0 0 1 [1]	Data. ${}^1V_2$ ○ 0 0 2 [2]	Data. ${}^1V_3$ ○ 0 0 4 [n]	Working Variables. ${}^0V_4$ ○ 0 0 0 [ ]	${}^0V_5$ ○ 0 0 0 [ ]	${}^0V_6$ ○ 0 0 0 [ ]	${}^0V_7$ ○ 0 0 0 [ ]	${}^0V_8$ ○ 0 0 0 [ ]	${}^0V_9$ ○ 0 0 0 [ ]	${}^0V_{10}$ ○ 0 0 0 [ ]	${}^0V_{11}$ ○ 0 0 0 [ ]	${}^0V_{12}$ ○ 0 0 0 [ ]	${}^0V_{13}$ ..... ○ 0 0 0 [ ]	Result Variables. ${}^1V_{21}$ B_1 in a decimal fraction. [B_1]	${}^1V_{22}$ B_3 in a decimal fraction. [B_3]	${}^1V_{23}$ B_5 in a decimal fraction. [B_5]	${}^0V_{24}$... ○ 0 0 0 [B_7]
1	×	${}^1V_2 \times {}^1V_3$	${}^1V_4, {}^1V_5, {}^1V_6$	${}^1V_2 = {}^1V_2$; ${}^1V_3 = {}^1V_3$	$= 2n$	...	2	n	$2n$	$2n$	$2n$											
2	−	${}^1V_4 - {}^1V_1$	${}^2V_4$	${}^1V_4 = {}^2V_4$; ${}^1V_1 = {}^1V_1$	$= 2n-1$	1	...	...	$2n-1$													
3	+	${}^1V_5 + {}^1V_1$	${}^2V_5$	${}^1V_5 = {}^2V_5$; ${}^1V_1 = {}^1V_1$	$= 2n+1$	1	...	...	...	$2n+1$												
4	÷	${}^2V_5 \div {}^2V_4$	${}^1V_{11}$	${}^2V_5 = {}^0V_5$; ${}^2V_4 = {}^0V_4$	$= \frac{2n-1}{2n+1}$	...	...	...	0	0	...	...	...	...	...	$\frac{2n-1}{2n+1}$						
5	÷	${}^1V_{11} \div {}^1V_2$	${}^2V_{11}$	${}^1V_{11} = {}^2V_{11}$; ${}^1V_2 = {}^1V_2$	$= \frac{1}{2} \cdot \frac{2n-1}{2n+1}$	...	2	...	...	...	...	...	...	...	...	$\frac{1}{2} \cdot \frac{2n-1}{2n+1}$						
6	−	${}^0V_{13} - {}^2V_{11}$	${}^1V_{13}$	${}^2V_{11} = {}^0V_{11}$; ${}^0V_{13} = {}^1V_{13}$	$= -\frac{1}{2} \cdot \frac{2n-1}{2n+1} = A_0$	...	...	...	...	...	...	...	...	...	...	0	.....	$-\frac{1}{2} \cdot \frac{2n-1}{2n+1} = A_0$				
7	−	${}^1V_3 - {}^1V_1$	${}^1V_{10}$	${}^1V_3 = {}^1V_3$; ${}^1V_1 = {}^1V_1$	$= n-1\ (=3)$	1	...	n	...	...	...	...	...	...	$n-1$							
8	+	${}^1V_2 + {}^0V_7$	${}^1V_7$	${}^1V_2 = {}^1V_2$; ${}^0V_7 = {}^1V_7$	$= 2 + 0 = 2$	...	2	...	...	...	...	2										
9	÷	${}^1V_6 \div {}^1V_7$	${}^3V_{11}$	${}^1V_6 = {}^1V_6$; ${}^0V_{11} = {}^3V_{11}$	$= \frac{2n}{2} = A_1$	...	...	...	...	...	$2n$	2	...	...	...	$\frac{2n}{2} = A_1$						
10	×	${}^1V_{21} \times {}^3V_{11}$	${}^1V_{12}$	${}^1V_{21} = {}^1V_{21}$; ${}^3V_{11} = {}^3V_{11}$	$= B_1 \cdot \frac{2n}{2} = B_1 A_1$	...	...	...	...	...	...	...	...	...	...	$\frac{2n}{2} = A_1$	$B_1 \cdot \frac{2n}{2} = B_1 A_1$	.....	B_1			
11	+	${}^1V_{12} + {}^1V_{13}$	${}^2V_{13}$	${}^1V_{12} = {}^0V_{12}$; ${}^1V_{13} = {}^2V_{13}$	$= -\frac{1}{2} \cdot \frac{2n-1}{2n+1} + B_1 \cdot \frac{2n}{2}$ ...	...	...	...	...	...	...	...	...	...	...	.....	0	$\left\{-\frac{1}{2} \cdot \frac{2n-1}{2n+1} + B_1 \cdot \frac{2n}{2}\right\}$				
12	−	${}^1V_{10} - {}^1V_1$	${}^2V_{10}$	${}^1V_{10} = {}^2V_{10}$; ${}^1V_1 = {}^1V_1$	$= n-2\ (=2)$	1	...	...	...	...	...	...	...	...	$n-2$							
13	−	${}^1V_6 - {}^1V_1$	${}^2V_6$	${}^1V_6 = {}^2V_6$; ${}^1V_1 = {}^1V_1$	$= 2n-1$	1	...	...	...	...	$2n-1$											
14	+	${}^1V_1 + {}^1V_7$	${}^2V_7$	${}^1V_1 = {}^1V_1$; ${}^1V_7 = {}^2V_7$	$= 2 + 1 = 3$	1	...	...	...	...	...	3										
15	÷	${}^2V_6 \div {}^2V_7$	${}^1V_8$	${}^2V_6 = {}^2V_6$; ${}^2V_7 = {}^2V_7$	$= \frac{2n-1}{3}$	...	...	...	...	...	$2n-1$	3	$\frac{2n-1}{3}$									
16	×	${}^1V_8 \times {}^3V_{11}$	${}^4V_{11}$	${}^1V_8 = {}^0V_8$; ${}^3V_{11} = {}^4V_{11}$	$= \frac{2n}{2} \cdot \frac{2n-1}{3}$	...	...	...	...	...	...	...	0	...	...	$\frac{2n}{2} \cdot \frac{2n-1}{3}$						
17	−	${}^2V_6 - {}^1V_1$	${}^3V_6$	${}^2V_6 = {}^3V_6$; ${}^1V_1 = {}^1V_1$	$= 2n-2$	1	...	...	...	...	$2n-2$											
18	+	${}^1V_1 + {}^2V_7$	${}^3V_7$	${}^2V_7 = {}^3V_7$; ${}^1V_1 = {}^1V_1$	$= 3 + 1 = 4$	1	...	...	...	...	...	4										
19	÷	${}^3V_6 \div {}^3V_7$	${}^1V_9$	${}^3V_6 = {}^3V_6$; ${}^3V_7 = {}^3V_7$	$= \frac{2n-2}{4}$	...	...	...	...	...	$2n-2$	4	...	$\frac{2n-2}{4}$	...	$\left\{\frac{2n}{2} \cdot \frac{2n-1}{3} \cdot \frac{2n-2}{3} = A_3\right\}$						
20	×	${}^1V_9 \times {}^4V_{11}$	${}^5V_{11}$	${}^1V_9 = {}^0V_9$; ${}^4V_{11} = {}^5V_{11}$	$= \frac{2n}{2} \cdot \frac{2n-1}{3} \cdot \frac{2n-2}{4} = A_3$	...	...	...	...	...	...	...	...	0								
21	×	${}^1V_{22} \times {}^5V_{11}$	${}^0V_{12}$	${}^1V_{22} = {}^1V_{22}$; ${}^0V_{12} = {}^2V_{12}$	$= B_3 \cdot \frac{2n}{2} \cdot \frac{2n-1}{3} \cdot \frac{2n-2}{3} = B_3 A_3$	...	...	...	...	...	...	...	...	...	...	0	$B_3 A_3$	.....	.....	B_3		
22	+	${}^2V_{12} + {}^2V_{13}$	${}^3V_{13}$	${}^2V_{12} = {}^0V_{12}$; ${}^2V_{13} = {}^3V_{13}$	$= A_0 + B_1 A_1 + B_3 A_3$	...	...	...	...	...	...	...	...	...	...	.....	0	$\left\{A_3 + B_1 A_1 + B_3 A_3\right\}$				
23	−	${}^2V_{10} - {}^1V_1$	${}^3V_{10}$	${}^2V_{10} = {}^3V_{10}$; ${}^1V_1 = {}^1V_1$	$= n-3\ (=1)$	1	...	...	...	...	...	...	...	...	$n-3$							

Here follows a repetition of Operations thirteen to twenty-three.

24	+	${}^4V_{13} + {}^0V_{24}$	${}^1V_{24}$	${}^4V_{13} = {}^0V_{13}$; ${}^0V_{24} = {}^1V_{24}$	$= B_7$	...	...	...	...	...	...	...	...	...	...	...	.....	.....	.....	.....	.....	.....	B_7
25	+	${}^1V_1 + {}^1V_3$	${}^1V_3$	${}^1V_1 = {}^1V_1$; ${}^1V_3 = {}^1V_3$; ${}^5V_6 = {}^0V_6$; ${}^5V_7 = {}^0V_7$	$= n+1 = 4+1 = 5$ ..... by a Variable-card. by a Variable card.	1	...	$n+1$	...	...	0	0											

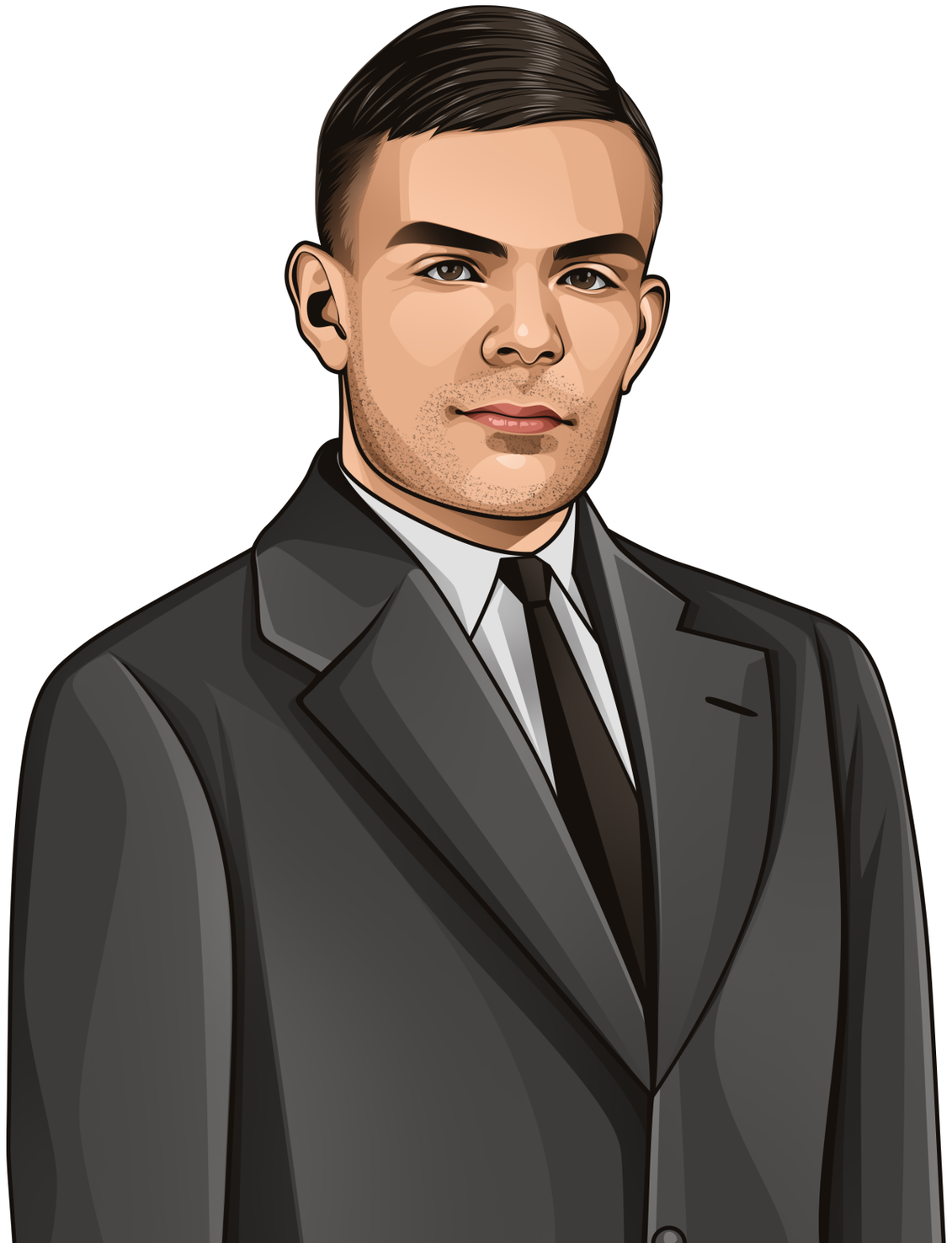
The Aftermath: Computers

The Aftermath: Computers

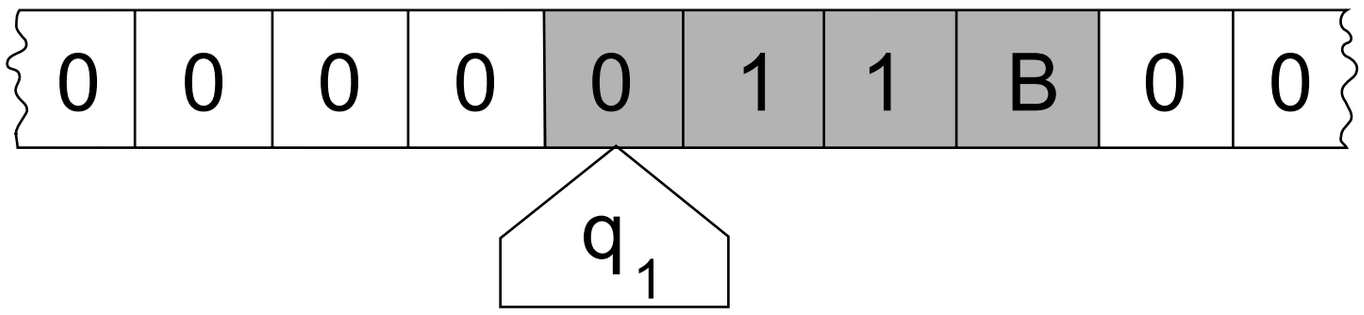


- The theoretical study of computers would be advanced in the 20th century by Alan Turing.

The Aftermath: Computers



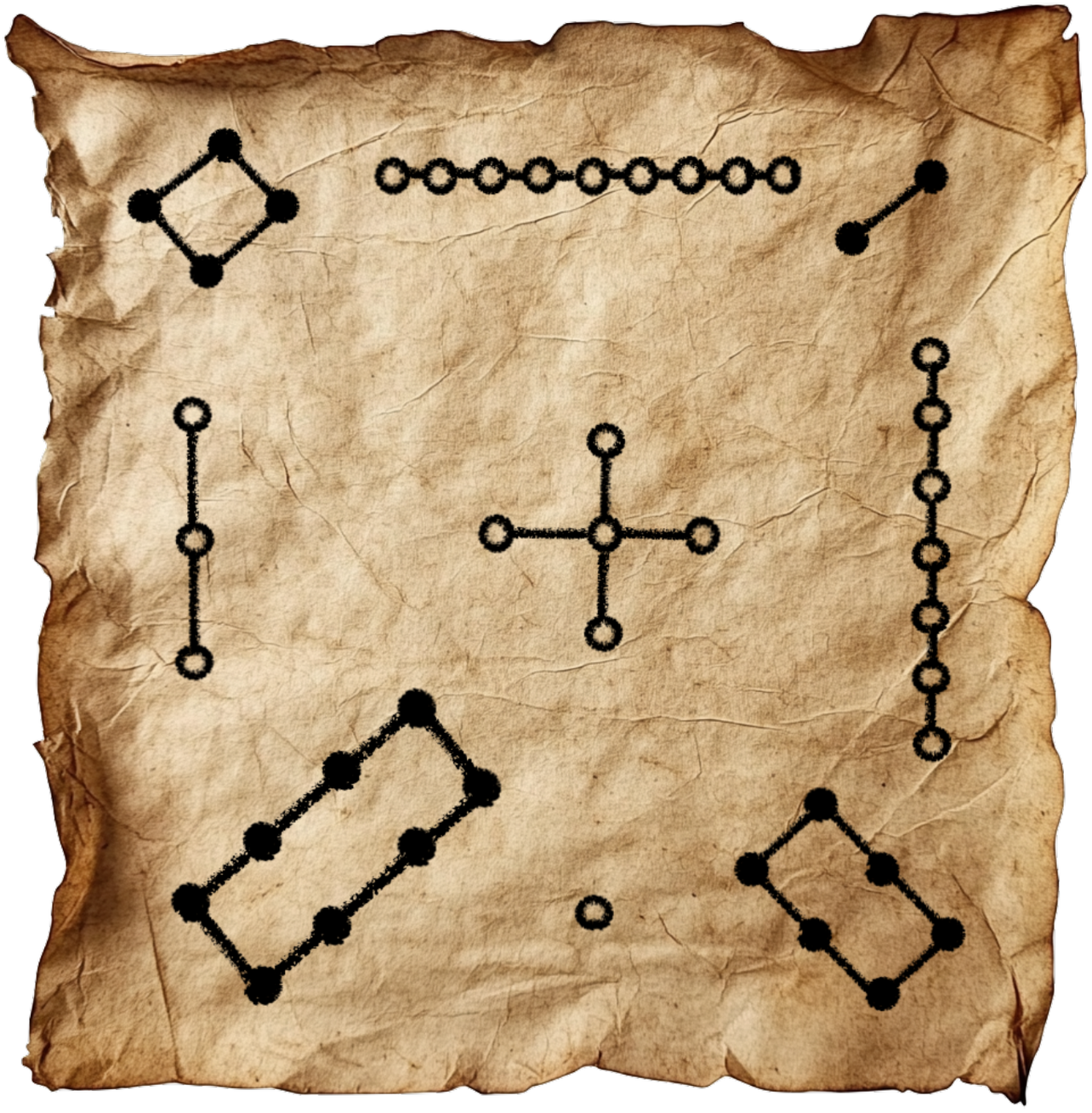
- The theoretical study of computers would be advanced in the 20th century by Alan Turing.
- To study a computer's limits, in 1936 he invented the "Turing machine."



Think Like A Math Historian

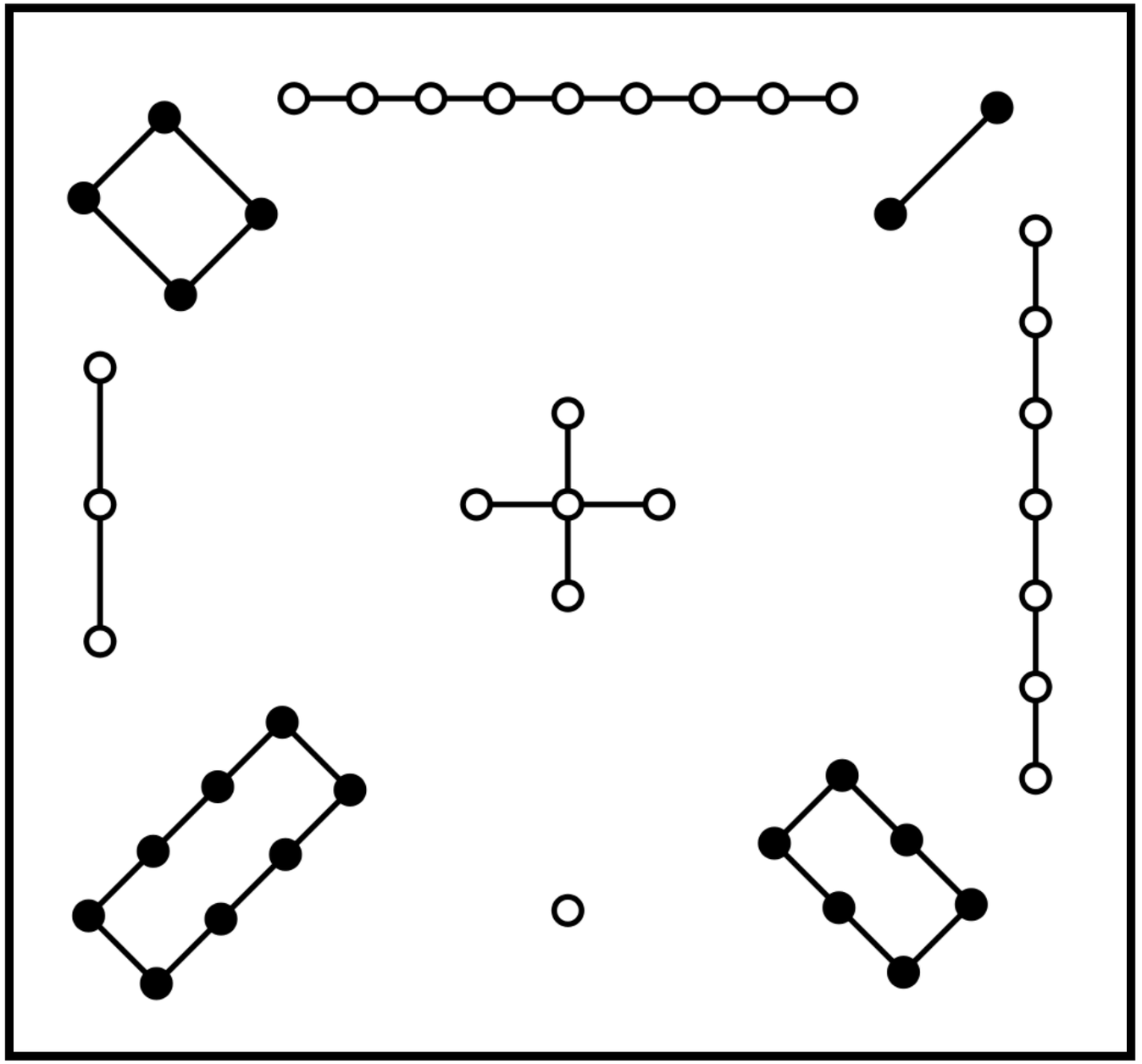
Think Like a Math Historian

- What does this mean?



Think Like a Math Historian

- Hint: Focus on how many dots are in each group.



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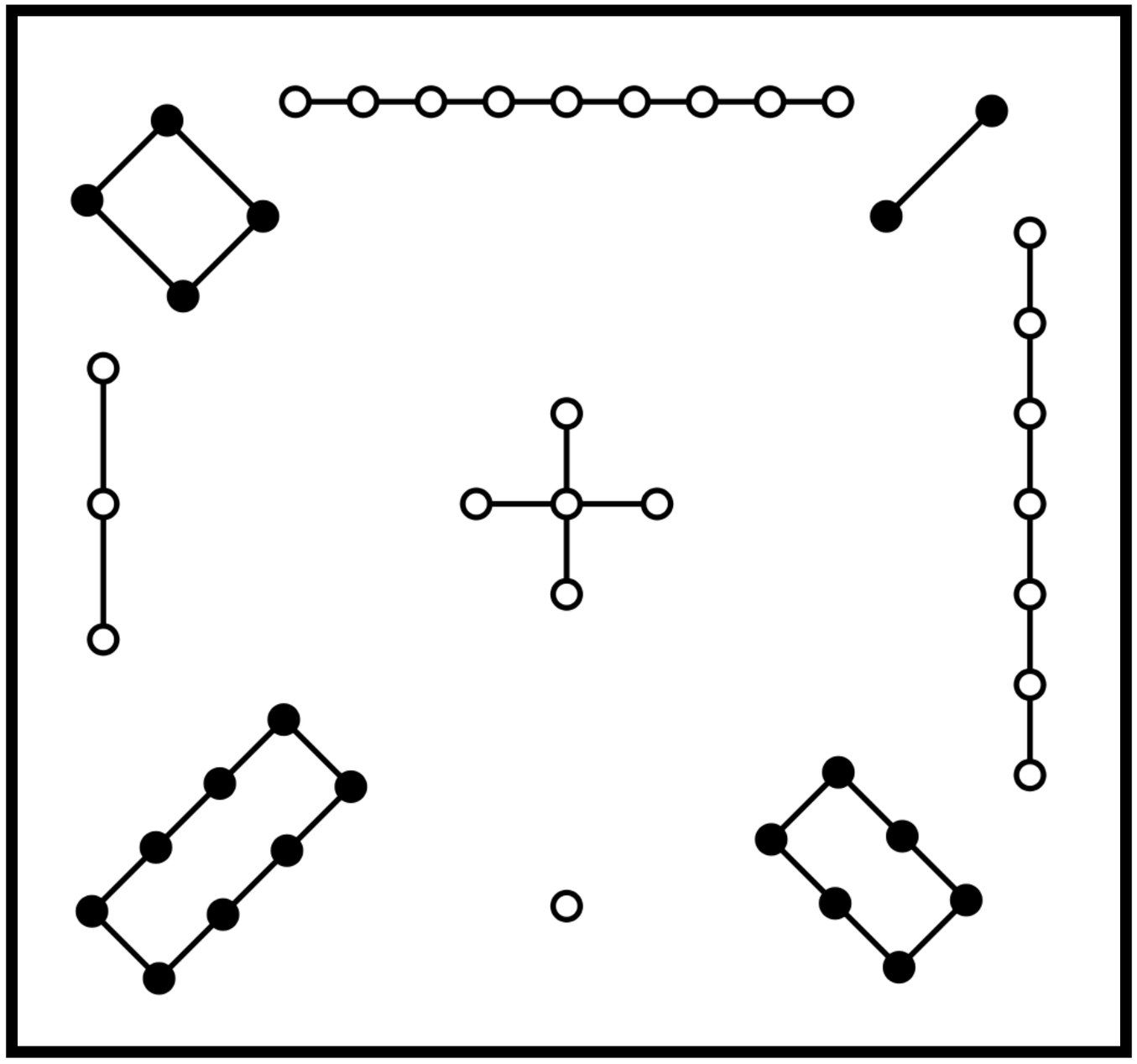

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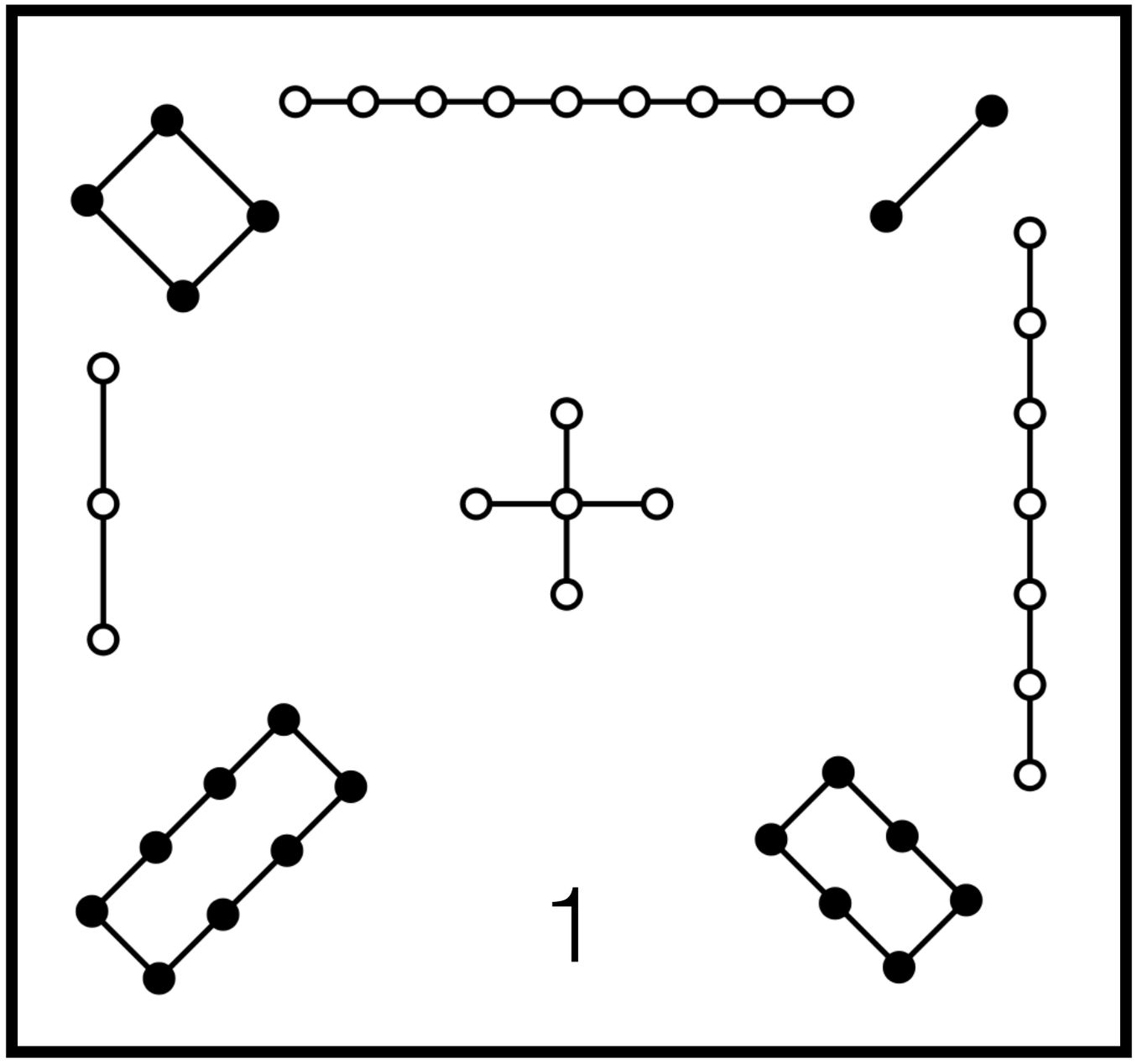

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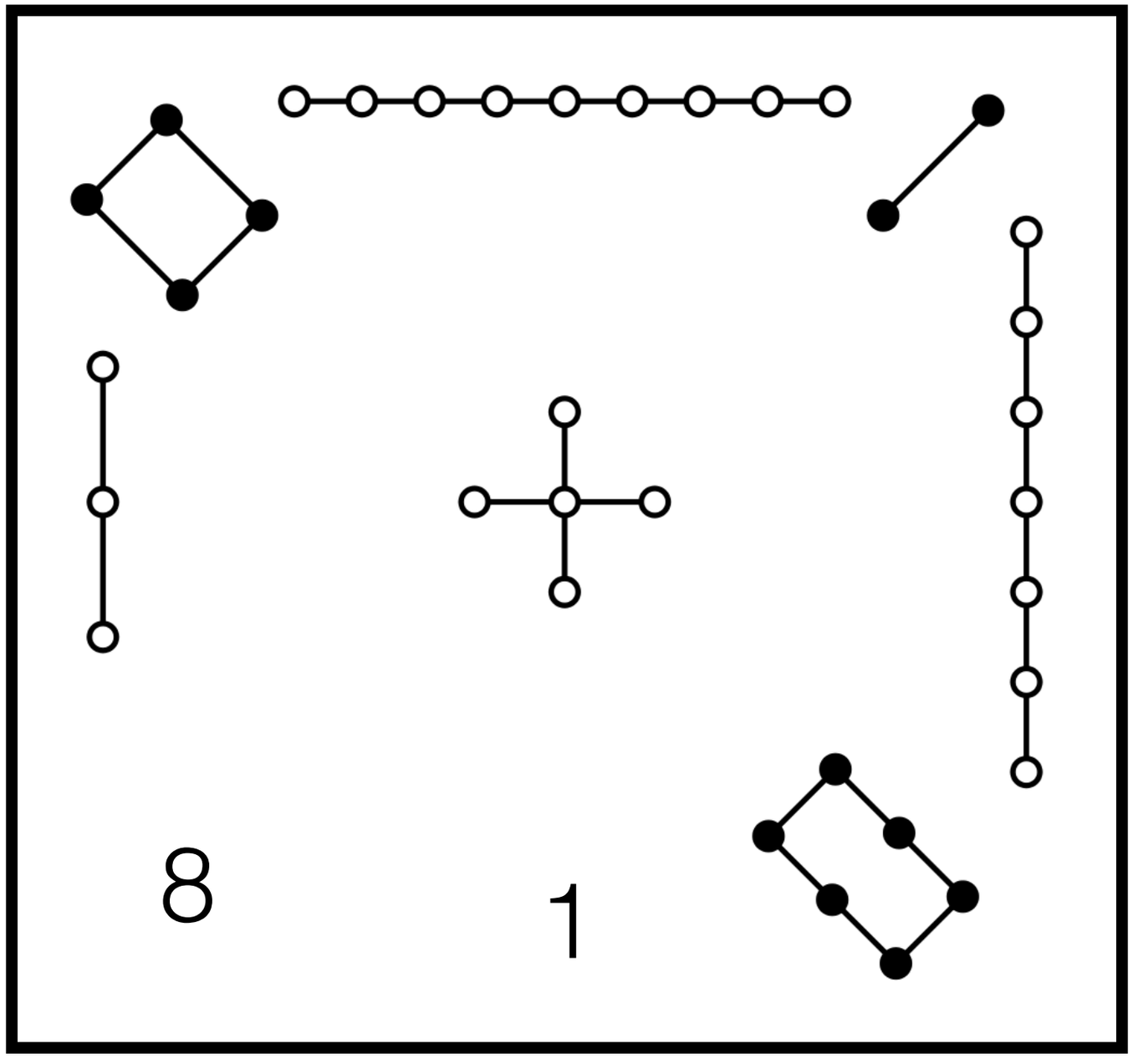

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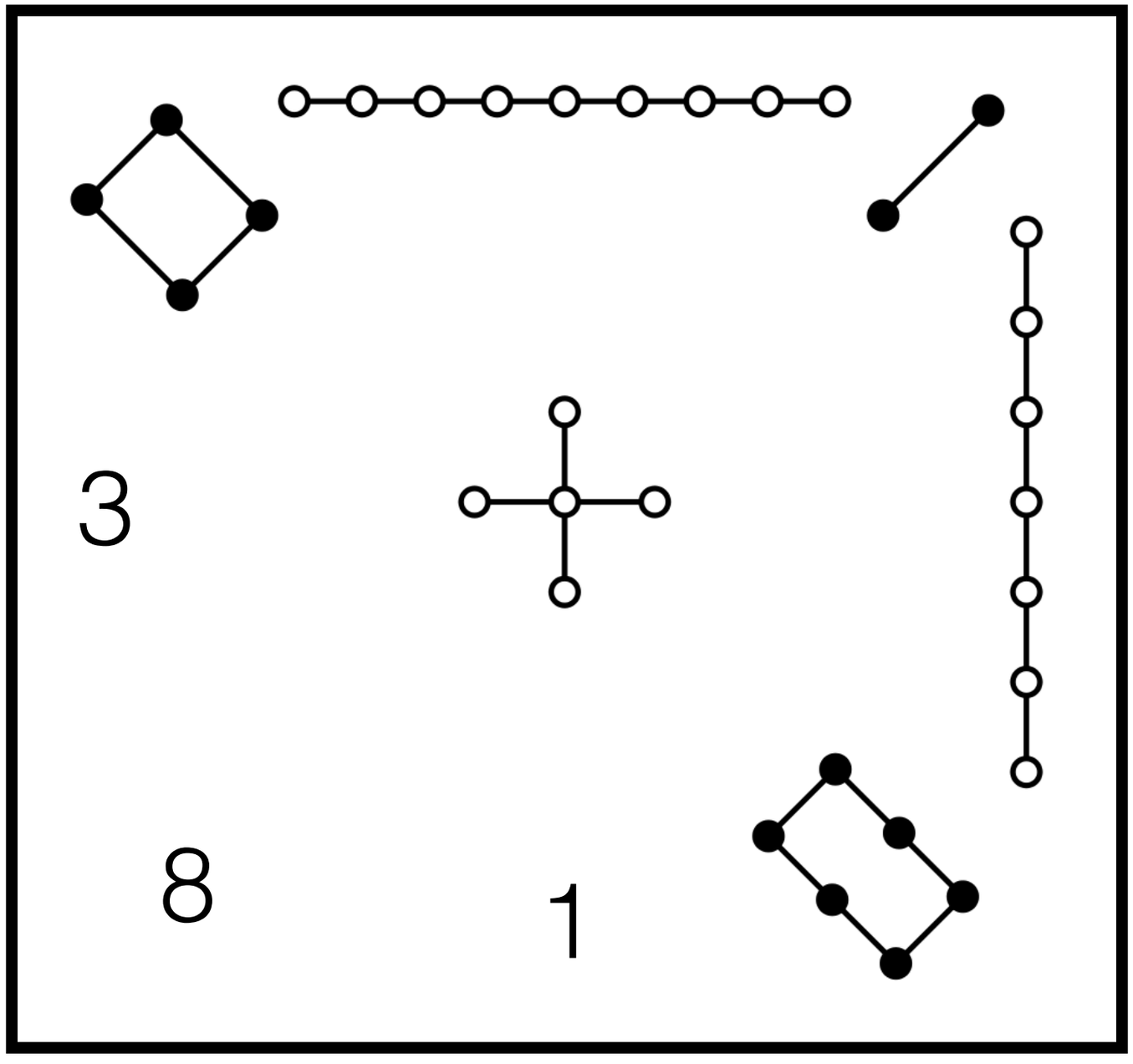

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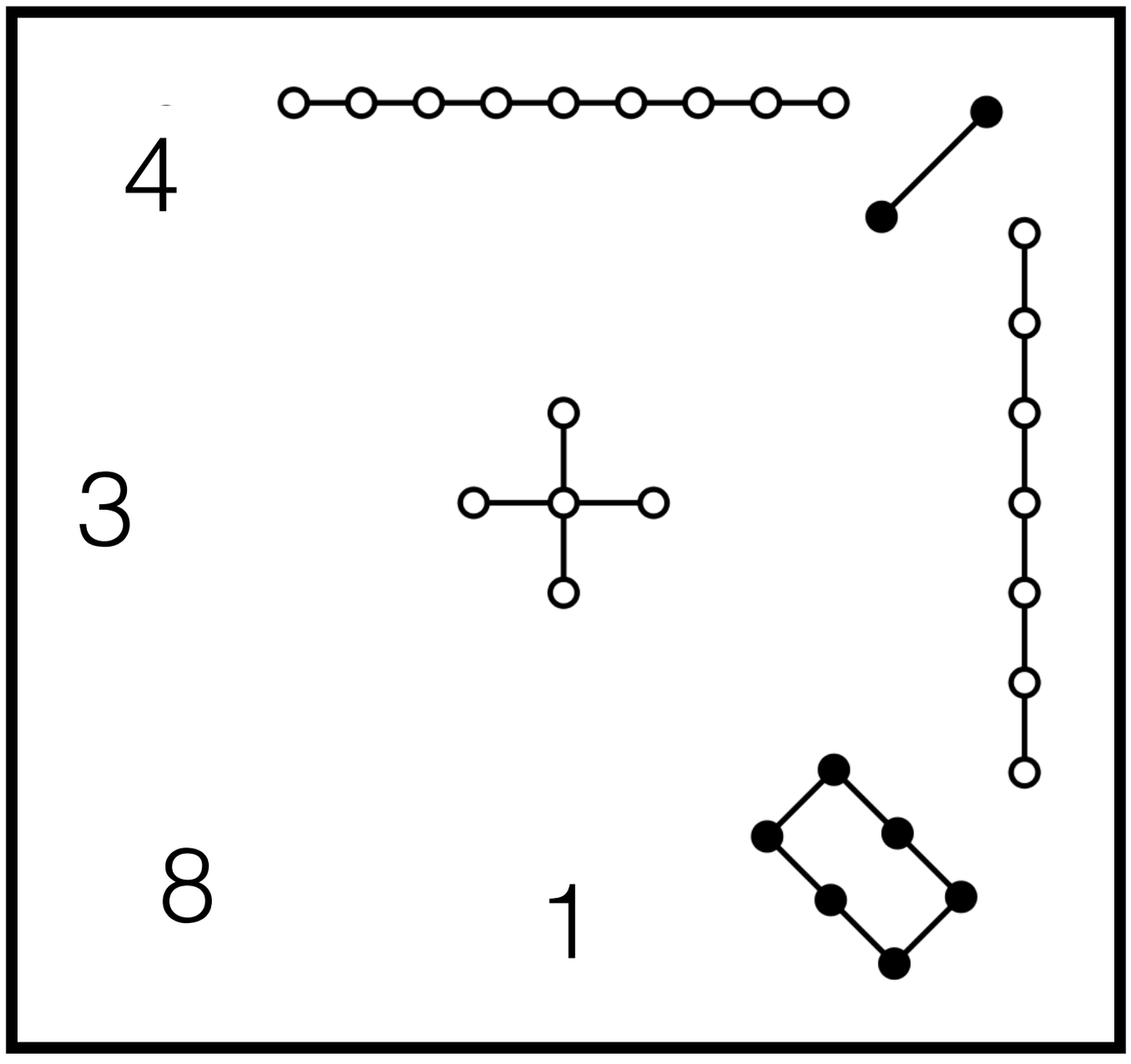

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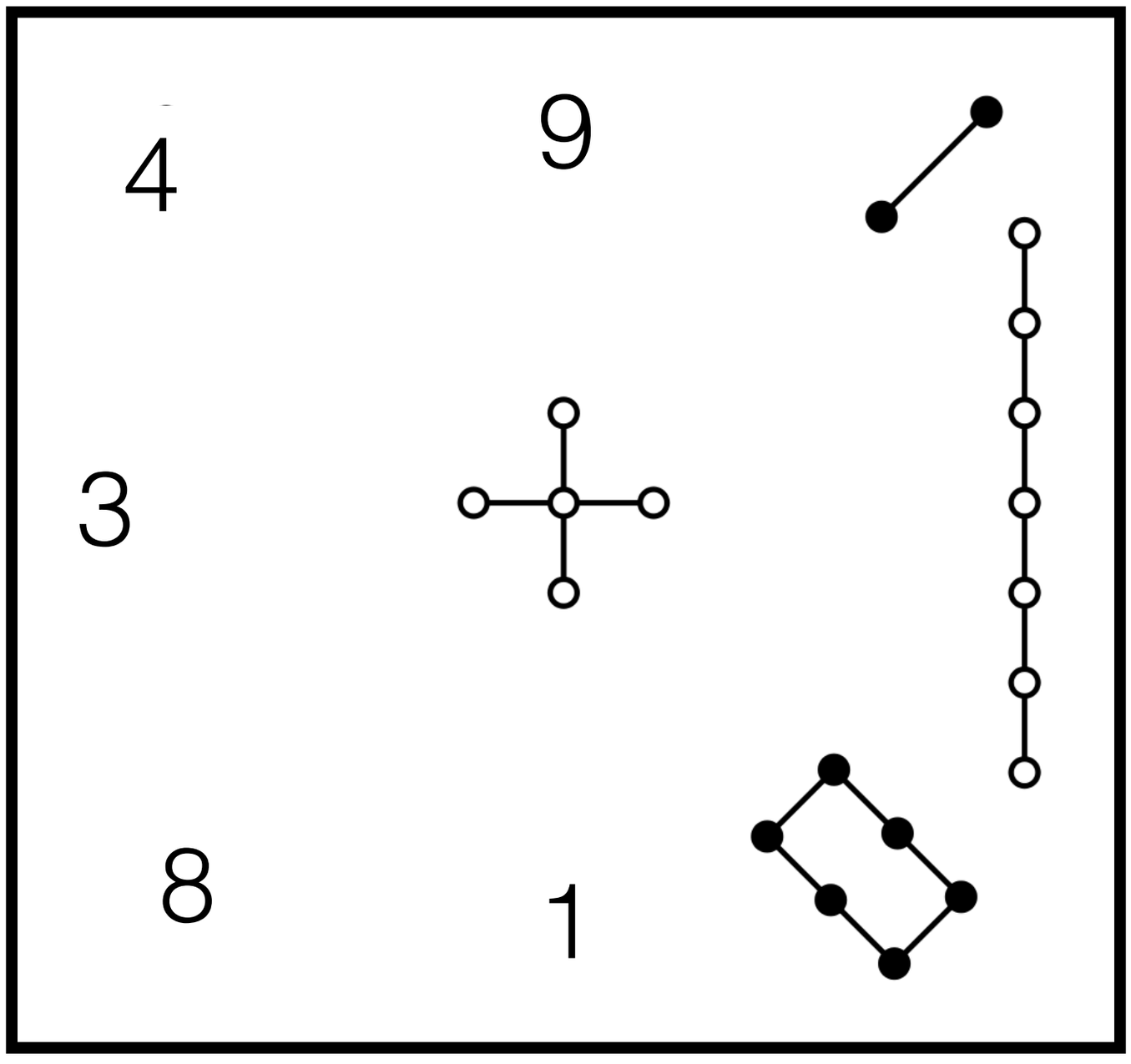

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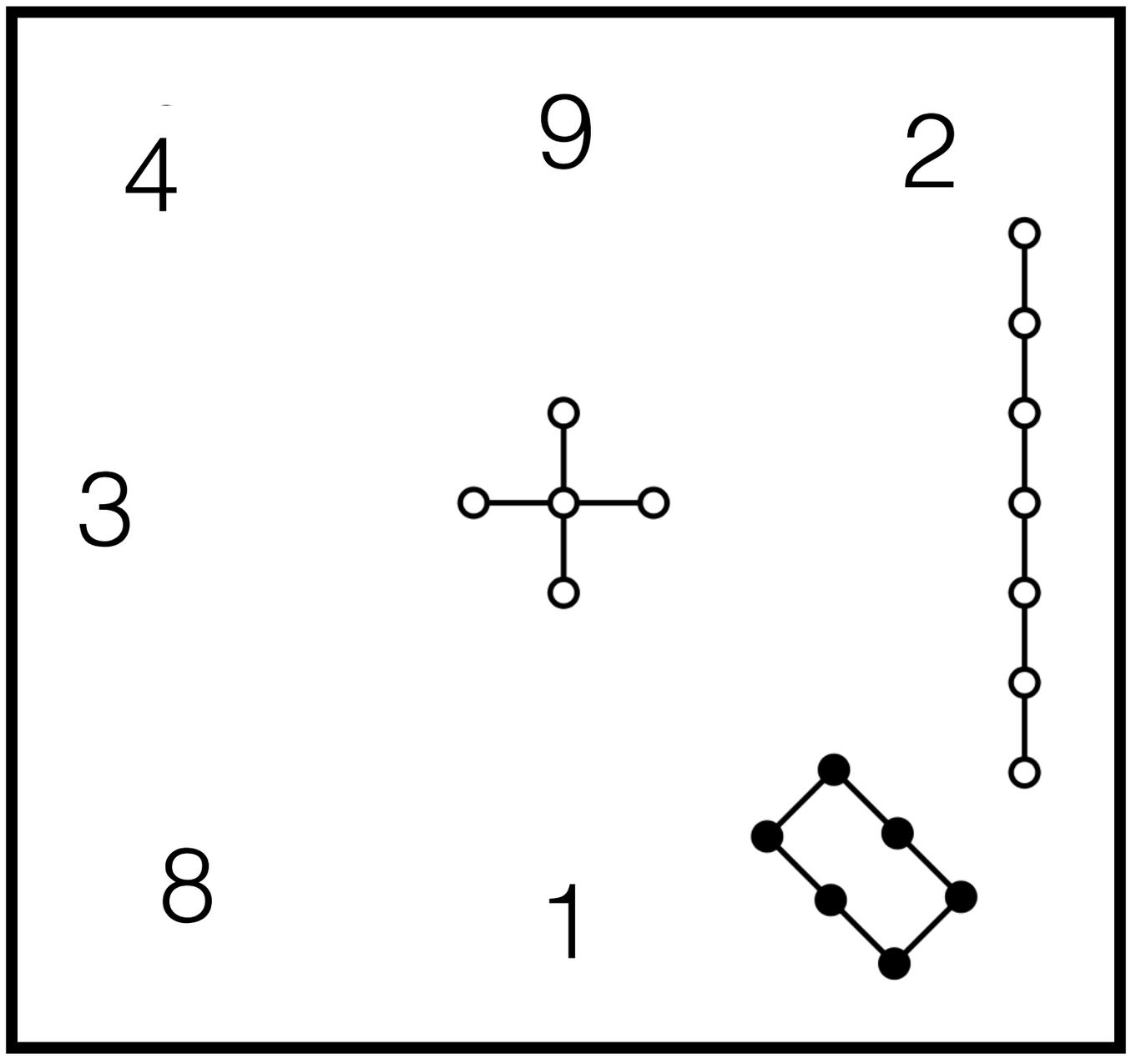

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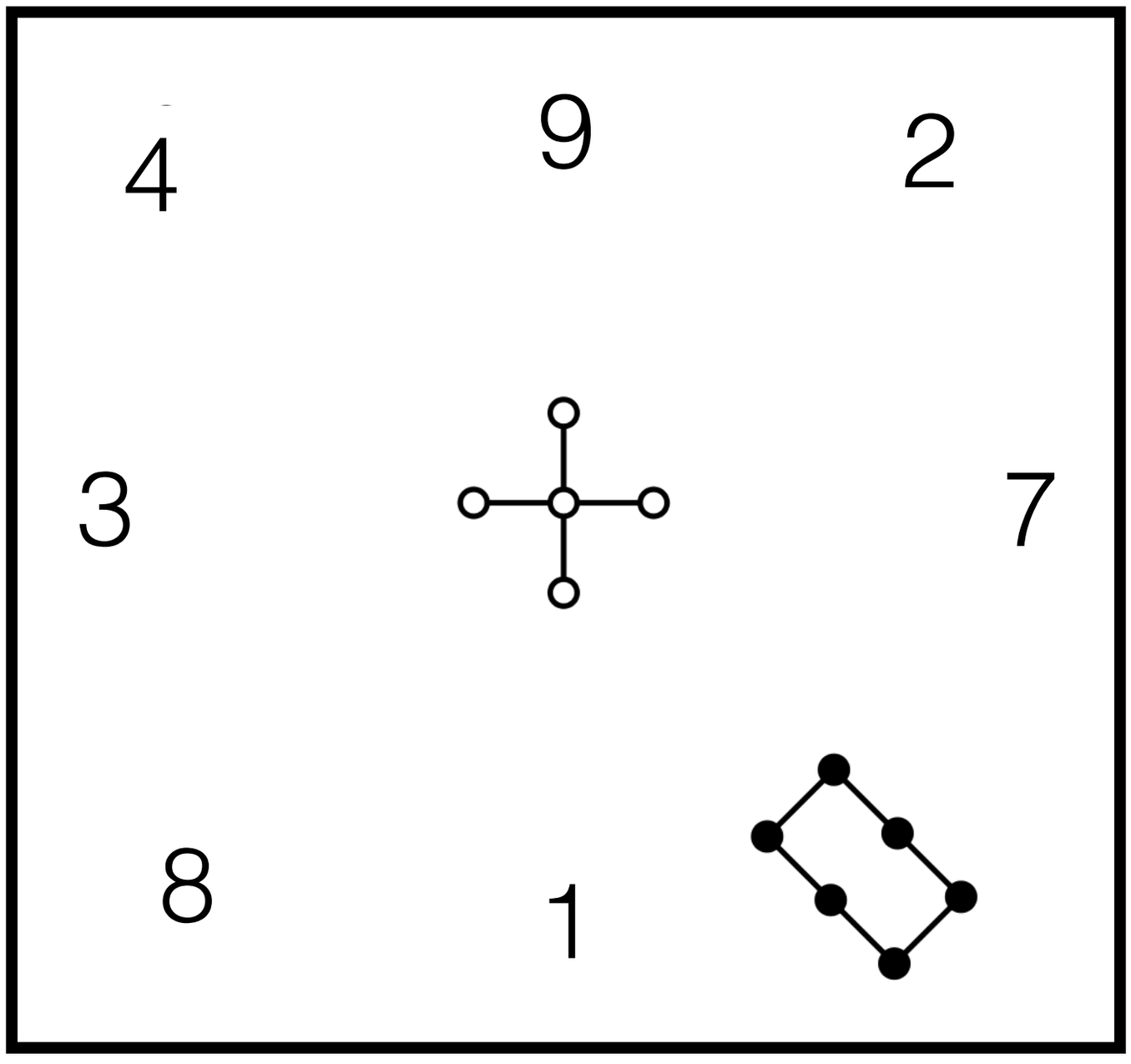

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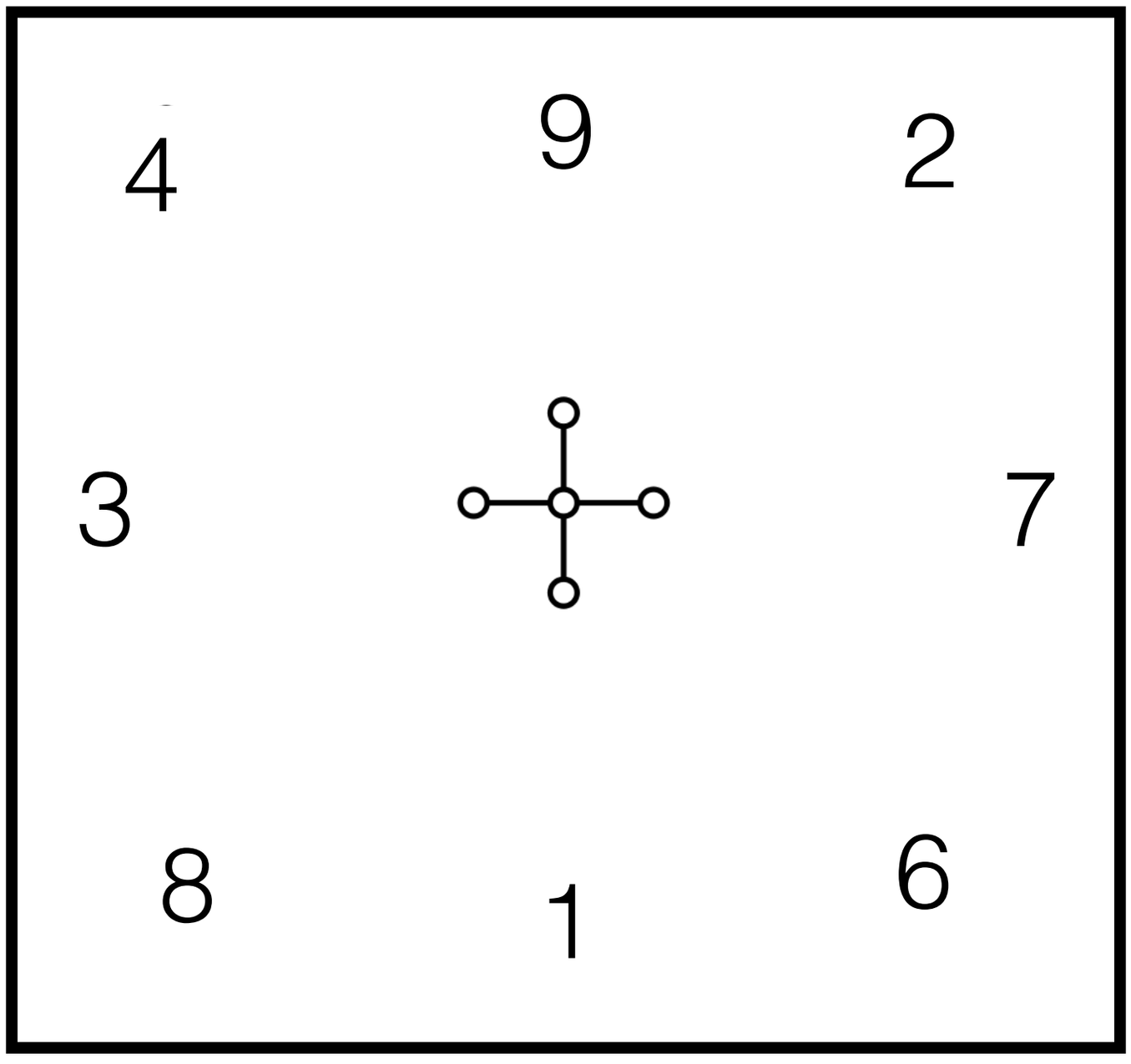

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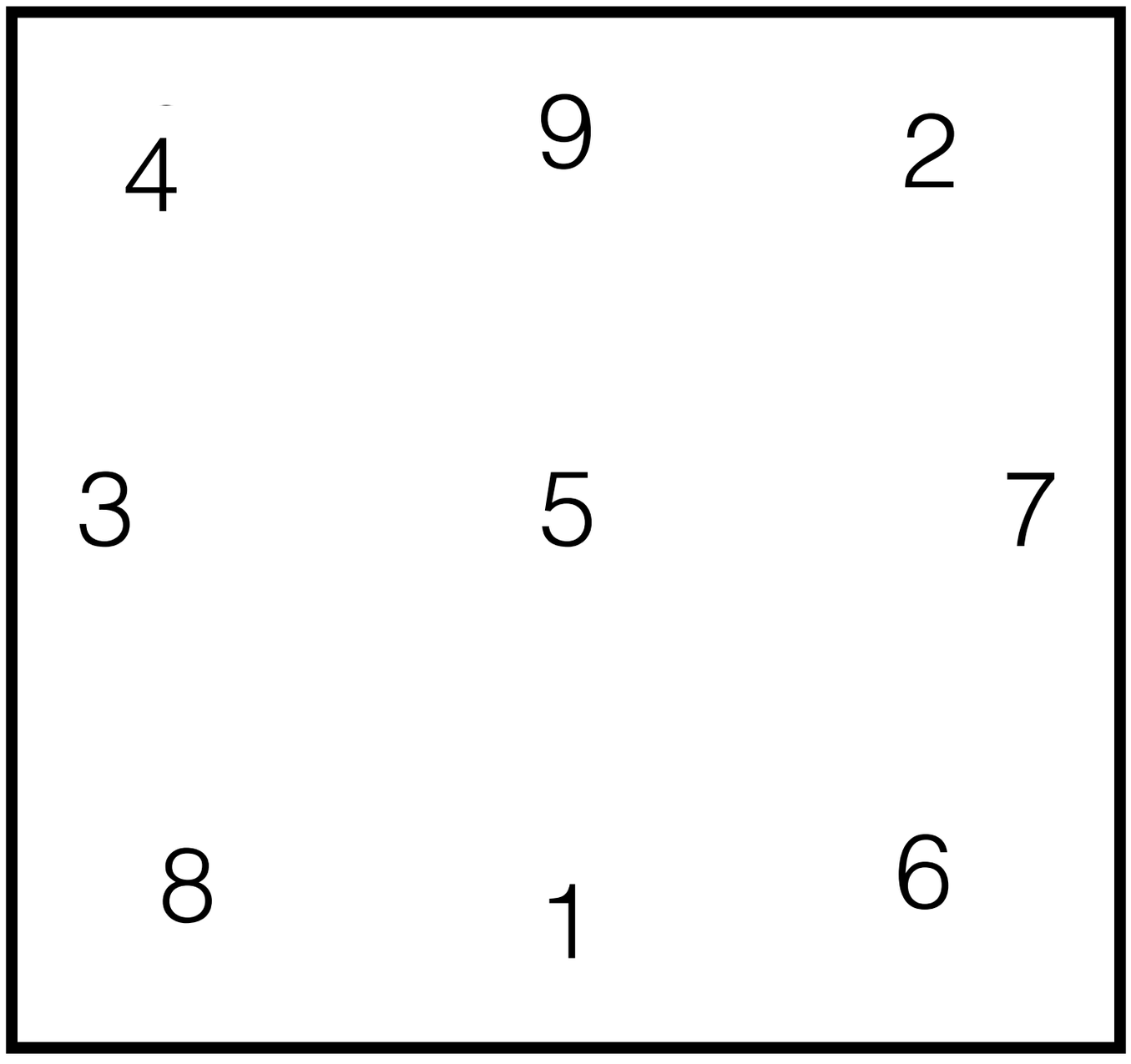

Think Like a Math Historian

- Magic Square

4	9	2
3	5	7
8	1	6

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The Aftermath

The Aftermath

- Yang Hui, a much more modern Chinese mathematician (~1238-1298 AD) had a 3-step procedure to create this square.

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<u>Step 1:</u> Write numbers in a diamond.

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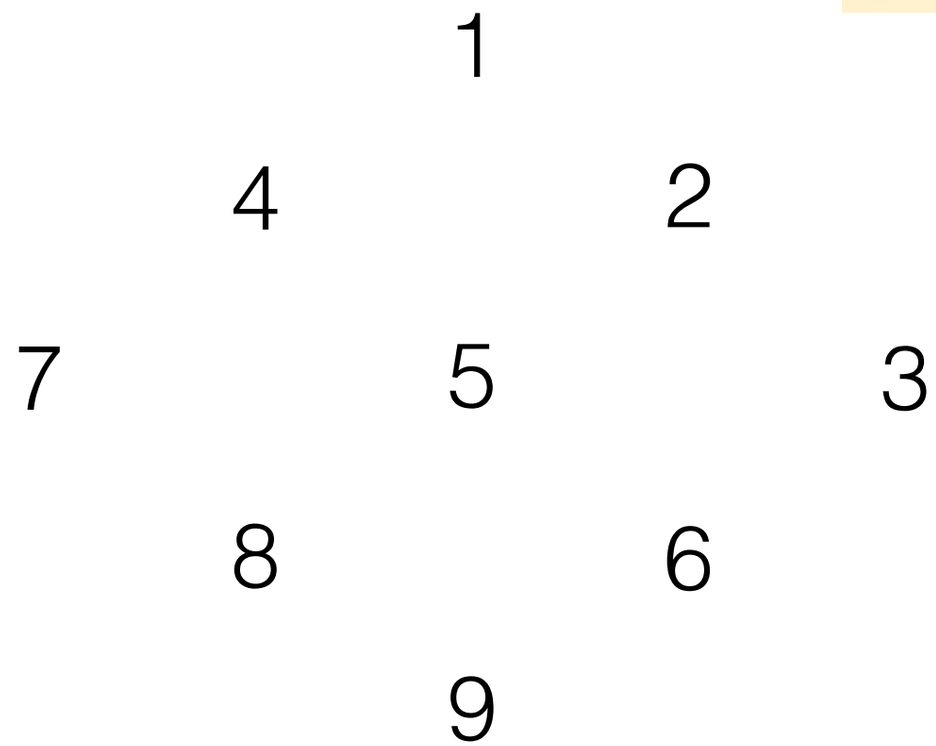

The Aftermath

- Yang Hui, a much more modern Chinese mathematician (~1238-1298 AD) had a 3-step procedure to create this square.

<u>Step 2:</u> Swap corners.

	1	
4		2
7	5	3
8		6
	9	

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<u>Step 2:</u> Swap corners.

	9	
4		2
7	5	3
8		6
	1	

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<u>Step 2:</u> Swap corners.

	9	
4		2
3	5	7
8		6
	1	

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<u>Step 3:</u> Scrunch together.

	9	
4		2
3	5	7
8		6
	1	

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4	9	2
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<u>Step 3:</u> Scrunch together.

4	9	2
3	5	7
8	1	6

Hui also came up with a similar algorithm to construct a 4x4 magic square.

The Aftermath

- Yang Hui constructed a magic square of every size up to 10x10.

The Aftermath

- Yang Hui constructed a magic square of every size up to 10x10.

一	二十	二一	四十	四一	六十	六一	八十	八一	一百
九九	八二	七九	六二	五九	四二	三九	二二	十九	二
三	十八	二三	三八	四三	五八	六三	七八	八三	九八
九七	八四	七七	六四	五七	四四	三七	二四	十七	四
五	十六	二五	三六	四五	五六	六五	七六	八五	九六
九五	八六	七五	六六	五五	四六	三五	二六	十五	六
十四	七	三四	二七	五四	四七	七四	六七	九四	八七
八八	九三	六八	七三	四八	五三	二八	三三	八	十三
十二	九	三二	二九	五二	四九	七二	六九	九二	八九
九一	九十	七一	七十	五一	五十	三一	三十	十一	十

The Aftermath

- Yang Hui constructed a magic square of every size up to 10x10.

一	二十	二一	四十	四一	六十	六一	八十	八一	一百
九九	八二	七九	六二	五九	四二	三九	二二	十九	二
三	十八	二三	三八	四三	五八	六三	七八	八三	九八
九七	八四	七七	六四	五七	四四	三七	二四	十七	四
五	十六	二五	三六	四五	五六	六五	七六	八五	九六
九五	八六	七五	六六	五五	四六	三五	二六	十五	六
十四	七	三四	二七	五四	四七	七四	六七	九四	八七
八八	九三	六八	七三	四八	五三	二八	三三	八	十三
十二	九	三二	二九	五二	四九	七二	六九	九二	八九
九一	九十	七一	七十	五一	五十	三一	三十	十一	十

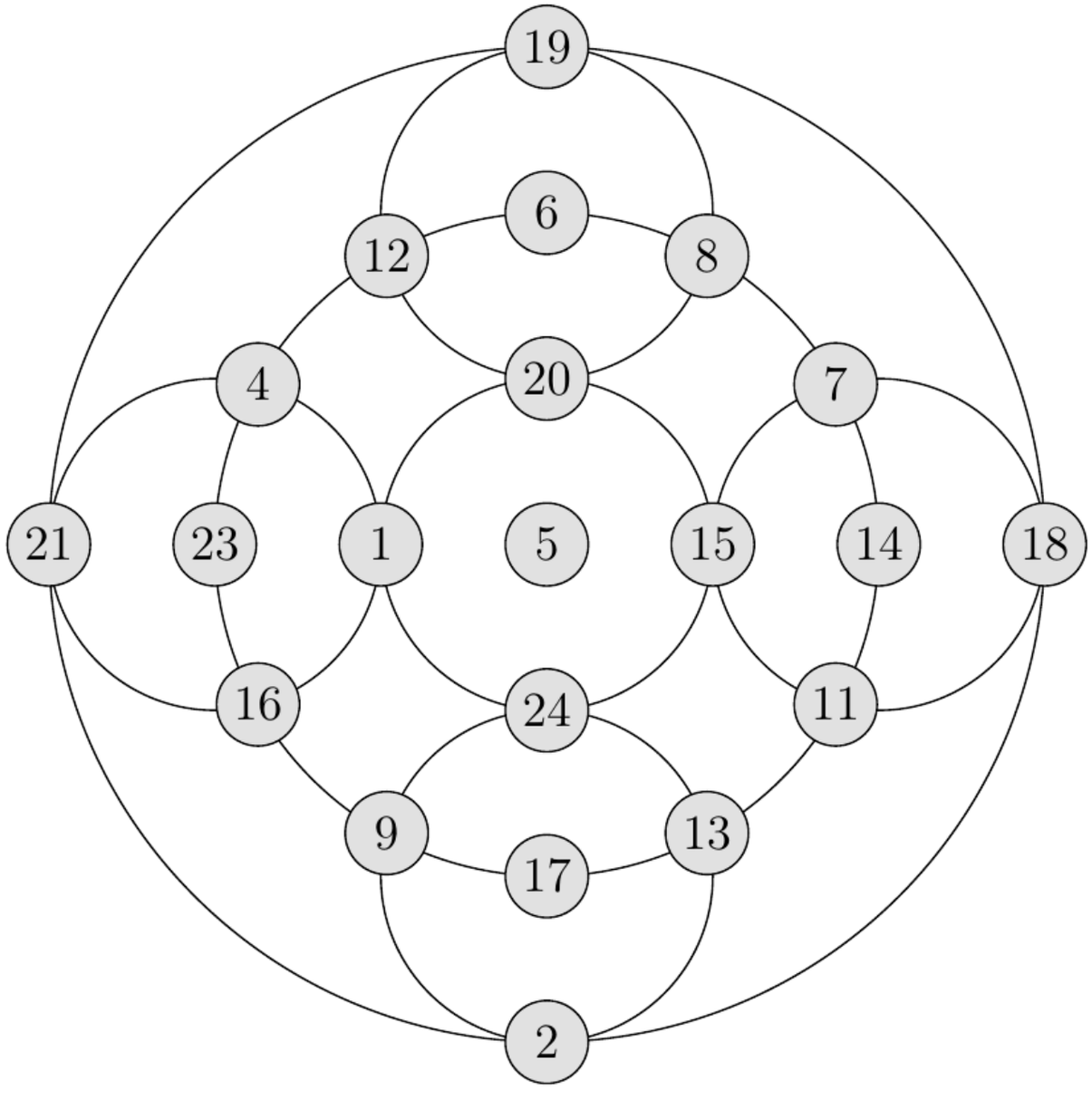
1	20	21	40	41	60	61	80	81	100
99	82	79	62	59	42	39	22	19	2
3	18	23	38	43	58	63	78	83	98
97	84	77	64	57	44	37	24	17	4
5	16	25	36	45	56	65	76	85	96
95	86	75	66	55	46	35	26	15	6
14	7	34	27	54	47	74	67	94	87
88	93	68	73	48	53	28	33	8	13
12	9	32	29	52	49	72	69	92	89
91	90	71	70	51	50	31	30	11	10

The Aftermath

- Yang Hui also constructed six magic circles.

The Aftermath

- Yang Hui also constructed six magic circles.



Algorithm to Construct Odd Magic Squares

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- Place a 1 in the top middle.
- Each time, move up and right, wrapping around when needed, writing the next number.

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Chapter 1

Shout-Outs

Counting Crows

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- Other animals may count too, to some extent. For example, crows.

Counting Crows

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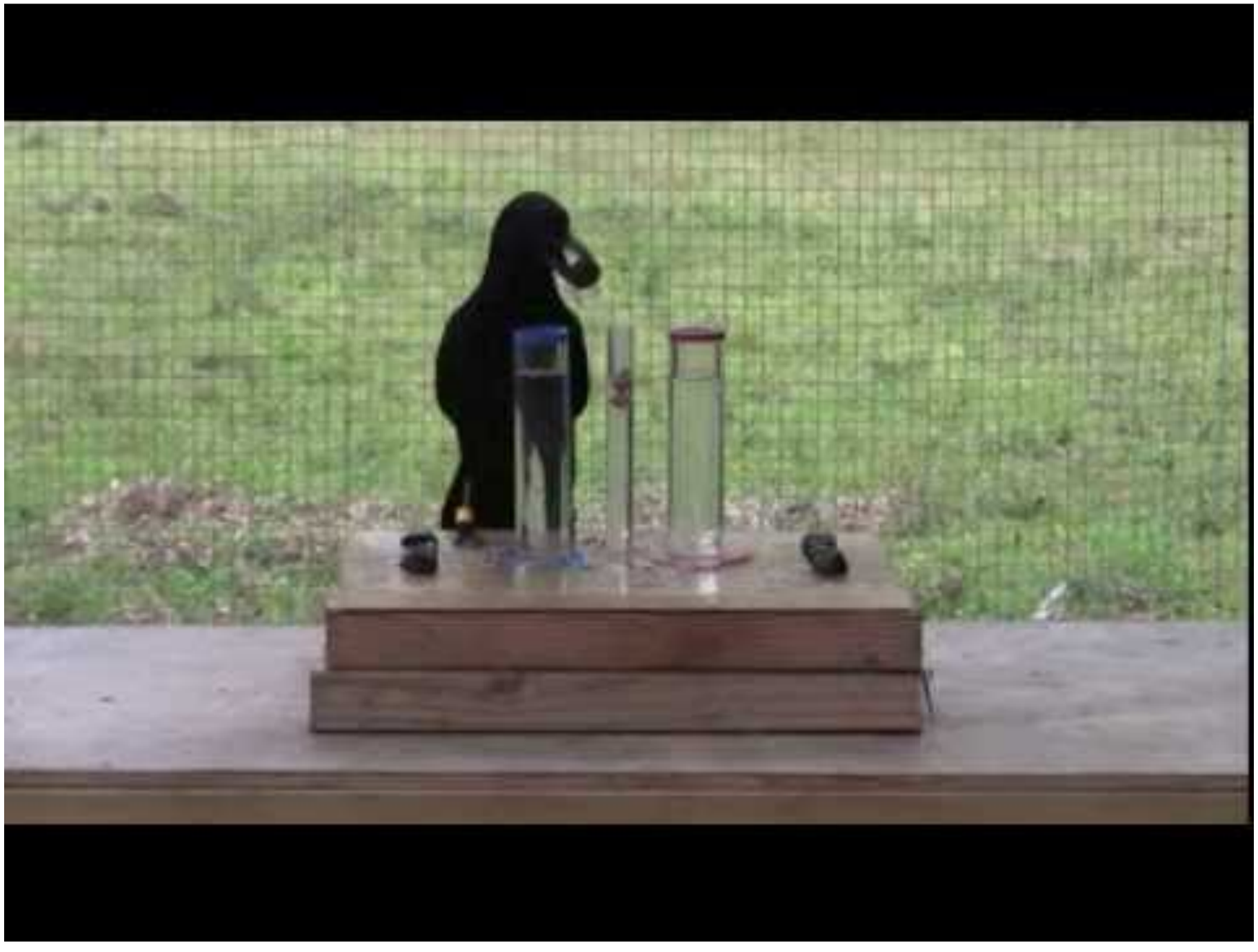
- Old story of a hunter hiding in a shelter.

Video: Counting Crows



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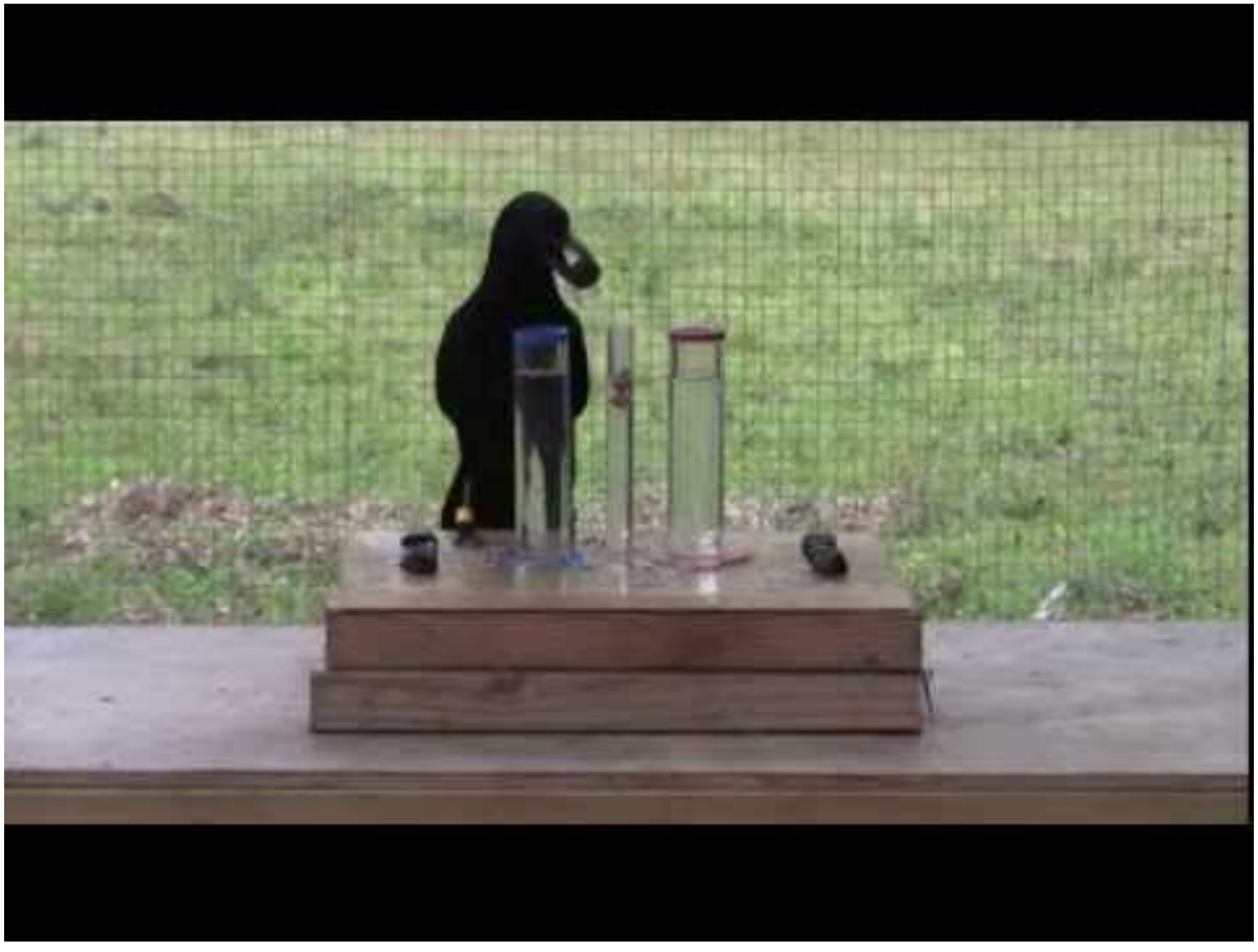

Napier’s Bones Shout-Out

- Napier’s Bones

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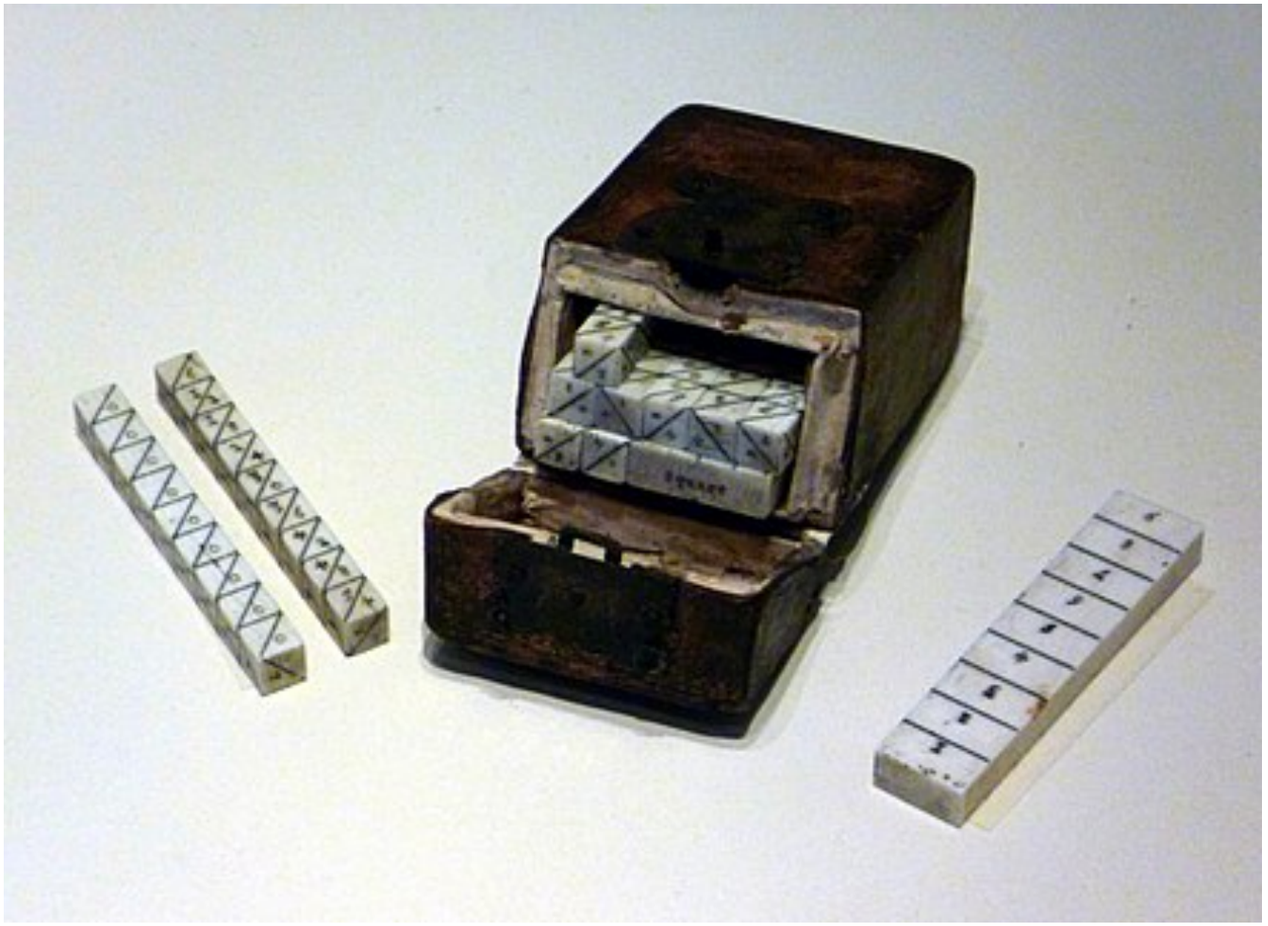

From the Appendices

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- Appendix B: The correct pronunciations of Charles Babbage, Ada Lovelace, John Napier, Blaise Pascal, Alan Turing, Acharya Virasena, Yang Hui.

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- Appendix F: The biographies of Ada Lovelace and Alan Turing.

People’s History

A People’s History of Math

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  - Two discuss ways that big social forces have influenced math history.
  - Two focus on the progression to the point where many people can choose to study math.
  - And one on who mathematicians were as people.

People’s History of Numbers

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- Later, these were often stored in boxes, and one might write on the box to label what was inside. This advancement is due to "the people."

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"It is India that gave us the ingenious method of expressing all numbers by means of ten symbols, each symbol receiving a value of position as well as an absolute value; a profound and important idea which appears so simple to us now that we ignore its true merit. But its very simplicity and the great ease which it has lent to all computations put our arithmetic in the first rank of useful inventions; and we shall appreciate the grandeur of this achievement the more when we remember that it escaped the genius of Archimedes and Apollonius, two of the greatest men produced by antiquity."

—Pierre-Simon Laplace (1749-1827)

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> The transition [in Europe to Hindu-Arabic numerals], far from being immediate, extended over long centuries. The struggle between the *Abacists*, who defended the old traditions, and the *Algorists*, who advocated the reform, lasted from the eleventh to the fifteenth century and went through all the usual stages of obscurantism and reaction. In some places, Arabic numerals were banned from official documents; in others, the art was prohibited altogether. And, as usual, did not succeed in abolishing, but merely served to spread *bootlegging*, ample evidence of which is found in the thirteenth century archives of Italy, where, it appears, merchants were using the Arabic numerals as a sort of secret code.
>
> —Tony Dantzig in *Number: The Language of Science*

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- Today, the math used in finance is incredibly advanced and quants are paid well to push their company's abilities further.